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ORIGINAL ARTICLE

SH plane-wave reflection/transmission coefficients in isotropic, attenuating media

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Abstract An attempt is made to extend the applicability of the weak-attenuation concept (WAC) to ray-theory computations. WAC allows an approximate evaluation of effects of attenuation on seismic-wave propagation in smoothly varying laterally inhomogeneous media encountered in most seismological studies. The goal is to find under which conditions WAC could be applicable to layered media. Specifically, if the use of WAC is necessary, under which conditions it can be used for the approximate evaluation of the reflection and transmission coefficients at interfaces separating attenuating media. It turns out that outside critical incidence regions, where the ray theory is not applicable, the effects of attenuation on the reflection and transmission are negligible in comparison with effects of attenuation on wave propagation inside layers. Despite it, the approximate formulae for reflection and transmission coefficients including effects of attenuation are derived and presented. For better insight and simplicity, effects of attenuation on SH plane-wave coefficients at interfaces separating homogeneous, attenuating isotropic half-spaces are studied. The coefficients are expressed in the form of the sum of coefficients in a reference elastic medium plus a perturbation due to weak attenuation. The study is based on the assumption of the validity of the correspondence principle despite indications of its inapplicability in some situations. A fixed frequency is considered. A basic role in the evaluation of coefficients is played by slowness vectors of incident and transmitted waves. They are required to satisfy constraints resulting from the corresponding equation of motion, Snell's law and radiation condition. The resulting formulae for coefficients are singular for the angles corresponding to critical incidence in the reference elastic medium. It is shown that the approximate formulae work well in the subcriti-

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cal region. Problems arise in the overcritical region of reference elastic media. The problems are related to the inapplicability of the commonly used correspondence principle. An artificial modification of formulae is proposed, which resolves the problem. However, it leads to the violation of the equation of motion and Snell's law constraints.

Keywords Attenuation · Weak-attenuation concept · Correspondence principle · SH-wave displacement reflection/transmission coefficients

1 Introduction

Realistic media are generally attenuating. A common way to study such media is to use the so-called *correspondence principle* (Bland 1960; Carcione 2014; Borcherdt 2009, 2020). This allows wave propagation results to be obtained in attenuating media from the results in the reference elastic media by replacing real-valued elastic parameters by their complex-valued anelastic counterparts. Although the conditions of applicability of the correspondence principle to the problem of reflection and transmission are in some cases not satisfied (Bland 1960, p.67; Borcherdt 2009, pp. 119–120), we use the correspondence principle as a basis of the approximate approach introduced below.

The motivation for our interest in evaluation of effects of attenuation on plane-wave displacement reflection and transmission coefficients is the generalization of program packages (Červený and Pšenčík 1984; Gajewski and Pšenčík 1990) based on high-frequency asymptotic methods, specifically the ray method, to laterally varying, layered attenuating media. In both packages, effects of attenuation can be considered inside layers using the so-called weak-attenuation concept (WAC). Although approximate, WAC is applicable in most seismological studies dealing with smooth media. Here, we study the effects of attenuation on reflection and transmission processes and necessity to consider them in the mentioned packages.

Straightforward application of the ray method to attenuating media leads to complex-valued ray tracing (Thomson 1997). This is a quite complicated and time-consuming procedure. If, however, attenuation is not too strong, which is often the case, it is possible to consider the attenuation as a perturbation of the elastic state, and use WAC. WAC was proposed by Kravtsov and Orlov (1990) and applied by Moczo et al. (1987) and Gajewski and Pšenčík (1992) to seismic wave propagation in smooth, isotropic and anisotropic, laterally varying, weakly attenuating media. Alternative approaches were proposed by, for example, Vavryčuk (2008) or Klimeš and Klimeš (2011). They differ in the way of the construction of the reference Hamiltonian used in the reference eikonal equation. All above-mentioned approaches lead to real-valued ray tracing in a reference elastic medium. Effects of attenuation are calculated by numerical quadratures along real-valued rays, which simplifies significantly the computational procedure.

We concentrate on the simplest case represented by reflection and transmission of an SH wave at an interface between two isotropic, attenuating media. This problem has been studied, among many others, by Buchen (1971), Borcherdt (1977), Krebes (1983), Richards (1984), Brokešová and Červený (1998), Ruud (2006), Krebes and Daley (2007), Sidler et al. (2008), Vavryčuk (2010), Daley and Krebes (2015) or Ursin et al. (2017). The problem is also discussed in several textbooks, see, e.g., Carcione (2014) or Borcherdt (2009, 2020).

Authors of many of the above-mentioned studies faced a problem of the choice of the sign of the square root in the expression for the vertical component of the slowness vector of a generated (transmitted) wave in case of overcritical incidence. The problem is described in a great detail by Krebes and Daley (2007) and investigated further by Daley and Krebes (2015). Some authors, for example Behura and Tsvankin (2009) or Sharma and Nain (2021), avoided this problem by concentrating on only subcritically reflected and transmitted waves or on waves reflected from a free surface. We are interested in both, subcritical and overcritical reflections, and try to avoid the problem of square roots by the use of the perturbation approach. Use of the perturbation approach implicitly includes "continuity" criterion of Krebes and Daley (2007), which should guarantee a continuous transition of the reflection or transmission coefficients from anelastic to elastic media when the attenuation diminishes.

The approximate approach proposed in this paper starts from the presentation of expressions for the plane-wave displacement SH reflection and transmission coefficients, which are formally the same for elastic and anelastic media. The only difference is that, due to the application of the correspondence principle, velocities and slowness vectors are real valued in elastic and complex valued in anelastic media (Krebes 1983; Carcione 2014; Krebes and Daley 2007; Borcherdt 2009). In this study, the complex-valued SH-wave slowness vectors are assumed to be confined to the plane of incidence. This guarantees the factorization of the system of equations for reflection and transmission coefficients into the 4×4 system for P and SV waves and 2×2 system for SH waves (Brokešová and Červený 1998).

In the following, we often use the terms subcritical, critical or overcritical incidence. It is important to emphasize here that they always relate to the reference elastic case.

The basic role in the expressions for the reflection and transmission coefficients is played by complex-valued slowness vectors of incident and transmitted waves. Real-valued parts of slowness vectors are called *propagation vectors* and imaginary parts are called *attenuation vectors*. If these two vectors are parallel, the corresponding wave is called *homogeneous*, if their directions differ, the wave is called *inhomogeneous*. The angle, which the two vectors make, is called the *attenuation (inhomogeneity) angle* γ. Slowness vectors of incident and generated homogeneous or inhomogeneous waves at an interface separating two weakly attenuating media should satisfy the following conditions:

a) the approximate constraint relation resulting from the corresponding equation of motion;
b) Snell's law, which requires equality of tangential components of complex-valued slowness vectors of the incident and generated waves;
c) the radiation condition, which requires decay of amplitudes of generated waves away from the interface in case of generation of inhomogeneous waves.

In WAC, attenuation represents a perturbation of the slowness vector in the reference elastic medium. The proposed approach does not involve the above-mentioned selection of signs of square roots determining vertical components of slowness vectors of transmitted wave in the anelastic medium. The problem of the selection of the sign is solved, in a standard way, in the reference elastic medium. Incident wave, generating reflected and transmitted waves may be homogeneous as well as inhomogeneous. The limitation of the presented approach is only its expected decrease of accuracy with increasing strength of attenuation and/or with increasing inhomogeneity of the incident wave. The resulting coefficients are singular for critical angles of waves incident at the interface in the reference elastic medium, and thus inaccurate in the vicinity of these angles (see Buchen 1971, discussing a similar problem). As Krebes and Daley (2007) point out, the singularity of the coefficients in the critical region is not a serious limitation since the ray theory does not work properly in the critical region.

The paper has the following structure. In the next section, we introduce the SH-wave reflection and transmission coefficients for attenuating media and describe the quantities specifying them. In Section 3, we specify slowness vectors of incident and transmitted waves. The behavior of slowness vectors of the transmitted wave in the transition from subcritical to overcritical region is analyzed. In Section 4, we discuss properties of the derived coefficients and present their simplified, slightly less accurate versions, in the form of "elastic" coefficients and their "anelastic" corrections. The accuracy of the derived expressions is then tested on several models in Section 5. We compare them with results obtained without the use of the correspondence principle (Daley and Krebes 2015). The main results are discussed in Section 6 and summarized in Section 7. The Einstein summation convention over repeated indices is used.

2 Basic equations

Expressions for the SH-wave displacement reflection and transmission coefficients at an interface separating two elastic half-spaces can be found in many forms in various textbooks (Červený, 2001, eq.5.3.2; Carcione 2014, eq.3.172; Aki and Richards 2002, eq.5.33; Chapman 2004, eq.6.3.7). Use of the correspondence principle (Carcione 2014; Borcherdt 2009, 2020) leads to the replacement of real-valued velocities in the formulae for reflection and transmission coefficients by their complex-valued counterparts. We consider SH waves whose propagation and attenuation vectors of incident and generated waves are situated in one plane, the plane of incidence (Borcherdt 2009, eqs 5.4.18, 5.4.19). We use the plane SH-wave reflection and transmission displacement coefficients in the form:

$$R = \frac{\rho_1 \beta_1^2 p_i N_i - \rho_2 \beta_2^2 p_k^{(t)} N_k}{\rho_1 \beta_1^2 p_i N_i + \rho_2 \beta_2^2 p_k^{(t)} N_k} ,$$
$$T = \frac{2\rho_1 \beta_1^2 p_i N_i}{\rho_1 \beta_1^2 p_i N_i + \rho_2 \beta_2^2 p_k^{(t)} N_k} . \tag{1}$$

Symbols ρ_1, ρ_2 and β_1, β_2 denote densities and S-wave velocities in the upper (ρ_1, β_1) and lower (ρ_2, β_2) half-spaces. Incident and reflected waves propagate in the upper half-space, transmitted wave propagates in the lower half-space. Symbols $\mathbf{p}$ and $\mathbf{p}^{(t)}$ denote slowness vectors of incident and transmitted SH waves, respectively. The symbol $\mathbf{N}$ in Eq. 1 denotes the unit normal to the interface. It is positive upwards, opposite to the direction of propagation of the incident wave.

In media with high quality factors Q, velocities β_m $(m = 1, 2)$ can be expressed approximately (neglecting higher-order terms) as in, e.g., Aki and Richards (2002), Krebes and Daley (2007), or Borcherdt (2009, 2020):

$$\beta_m(\omega) = \beta_m^R(\omega) + \mathrm{i}\beta_m^I(\omega) = \beta_m^R(\omega)[1 - \frac{\mathrm{i}}{2} Q_m^{-1}(\omega)] \tag{2}$$

(no summation over m). Symbol Q_m denotes the quality factor in the mth half-space. Throughout the paper, we work exclusively with the S-wave velocity β_m^R of the reference elastic medium. For the sake of simplicity, we denote it β_m in the following, without using the superscript R. Velocities β_m as well as quality factors Q_m are frequency dependent, i.e., $\beta_m = \beta_m(\omega)$, $Q_m = Q_m(\omega)$. We consider ω fixed throughout the paper. The use of the minus sign in on the right-hand side of Eq. 2 corresponds to the form of the exponential factor of the time-harmonic plane waves used: $\exp[-\mathrm{i}\omega(t - p_k x_k)]$. The components p_k and $p_k^{(t)}$ of slowness vectors of incident, $\mathbf{p}$, and transmitted, $\mathbf{p}^{(t)}$, waves in attenuating media can be expressed in the following form:

$$p_i = P_i + \mathrm{i}A_i , \qquad p_i^{(t)} = P_i^{(t)} + \mathrm{i}A_i^{(t)} . \tag{3}$$

In Eq. 3, $\mathbf{P}$ and $\mathbf{P}^{(t)}$ are propagation vectors, and $\mathbf{A}$ and $\mathbf{A}^{(t)}$ are attenuation vectors. Note that instead of the slowness vector $\mathbf{p}$, some authors (e.g., Aki and Richards 2002) decompose the wave vector $\mathbf{k} = \omega\mathbf{p}$. Propagation and attenuation vectors are real valued. For the incident wave, the propagation vector $\mathbf{P}$ is the slowness vector in the reference elastic medium, and the attenuation vector $\mathbf{A}$ represents its perturbation.

3 Slowness vectors

Under the weak-attenuation approximation, ($Q \gg 1$; higher-order terms containing Q^{-1} neglected), slowness vectors of incident and transmitted waves should satisfy important constraint relations resulting from the equation of motion in the corresponding attenuating medium:

$$p_i p_i = \beta^{-2}(1 + \mathrm{i}Q^{-1}) . \tag{4}$$

Inserting $\mathbf{p}$ and $\mathbf{p}^{(t)}$ from Eq. 3 to Eq. 4, and neglecting higher-order terms, we obtain:

$$P_i P_i = \beta_1^{-2} , \quad P_i A_i = \frac{1}{2}\beta_1^{-2} Q_1^{-1} ,$$
$$P_i^{(t)} P_i^{(t)} - A_i^{(t)} A_i^{(t)} = \beta_2^{-2} , \quad P_i^{(t)} A_i^{(t)} = \frac{1}{2}\beta_2^{-2} Q_2^{-1} . \tag{5}$$

Let us first specify the slowness vector of the incident wave.

3.1 Incident wave

If the incident wave propagates in an elastic medium, its attenuation vector is zero, $\mathbf{A} = 0$. In this case, components of the slowness vector read:

$$p_i = P_i . \tag{6}$$

In an attenuating medium, the attenuation vector $\mathbf{A}$ is non-zero. We consider it to be situated in the plane of incidence formed by vectors $\mathbf{P}$ and $\mathbf{N}$, and to be small in order to represent a perturbation of $\mathbf{P}$. The slowness vector $\mathbf{p}$ satisfying the first and second of Eq. 5 has components:

$$p_i = P_i + \mathrm{i}A_i = P_i + \frac{\mathrm{i}}{2} a_i \beta_1^{-1} Q_1^{-1} \cos^{-1}\gamma$$
$$= P_i + \mathrm{i}\Big(\frac{1}{2} Q_1^{-1} P_i + D m_i\Big) . \tag{7}$$

In Eq. 7, $\mathbf{a}$ is a unit vector in the direction of the attenuation vector $\mathbf{A}$, γ is the attenuation angle (the angle between $\mathbf{P}$ and $\mathbf{A}$). The vector $\mathbf{m}$ is a unit vector perpendicular to $\mathbf{P}$, situated in the plane of incidence. It is obtained by the clockwise rotation from the vector $\mathbf{P}$ in the plane of incidence. The slowness vector $\mathbf{p}$ is thus confined to the plane of incidence. The symbol D denotes the *inhomogeneity factor* (Červený and

Pšenčík 2008). For $D = 0$, the wave is homogeneous, for $D \neq 0$, the wave is inhomogeneous. The inhomogeneity factor D represents a more convenient characterization of the inhomogeneity of a wave than the attenuation angle γ. For the relation between the factor D and the attenuation angle γ for SH waves in an isotropic viscoelastic medium, see equations (7) and (12) of Červený and Pšenčík (2005).

In WAC, the slowness vector $\mathbf{p}$ whose components are given in Eq. 7 can be considered as the slowness vector in the reference elastic medium plus a perturbation due to the attenuation and/or inhomogeneity of the wave. The perturbation term is the attenuation vector $\mathbf{A}$. This means that $\beta_1^{-1} Q_1^{-1}$ and D must be small.

3.2 Transmitted wave

The complex-valued slowness vector of a transmitted wave, $\mathbf{p}^{(t)}$, should, in addition to the approximate satisfaction of the constraint relation (4) in the lower half-space, satisfy generalized Snell's law:

$$p_i^{(t)} - (p_k^{(t)} N_k) N_i = p_i - (p_k N_k) N_i \, . \tag{8}$$

Equation 8 implies that horizontal components of slowness vectors of the generated and incident waves are the same. The real part of the horizontal component of the slowness vector is the real-valued ray parameter p,

$$p = P_i - N_i (P_k N_k) = \sin i / \beta_1 \, . \tag{9}$$

The symbol i in Eq. 9 denotes the incidence angle. For $p_i^{(t)}$, we get from Eq. 8:

$$p_i^{(t)} = p_i - (p_k N_k) N_i + (p_k^{(t)} N_k) N_i \, , \tag{10}$$

see also equation (19) of Červený (2007). Our goal is the evaluation of the projection $p_k^{(t)} N_k$ of the slowness vector $\mathbf{p}^{(t)}$ of the transmitted wave to the normal $\mathbf{N}$ to the reflector.

Let us introduce the following notation:

$$\begin{aligned} X_1 &= -\beta_1 P_i N_i \, , \qquad & X_2 &= -\beta_2 P_i^{(t)} N_i \, , \\ \xi &= -\beta_1 A_i N_i \, , \qquad & \xi^{(t)} &= -\beta_2 A_i^{(t)} N_i \, . \end{aligned} \tag{11}$$

Using Eq. 3 and the notation in Eq. 11, we can rewrite Eq. 10 as

$$\begin{aligned} P_i^{(t)} + \mathrm{i} A_i^{(t)} = {} & P_i + \beta_1^{-1} X_1 N_i + \mathrm{i}(A_i + \beta_1^{-1} \xi N_i) \\ & - \beta_2^{-1} X_2 N_i - \mathrm{i} \beta_2^{-1} \xi^{(t)} N_i \, . \end{aligned} \tag{12}$$

The quantities X_1 and X_2 in Eq. 12 are the square roots, well known from studies of reflection and transmission in elastic media. They can be expressed in terms of the ray parameter p and the corresponding velocity β_m:

$$X_1 = (1 - \beta_1^2 p^2)^{1/2} \, , \qquad X_2 = (1 - \beta_2^2 p^2)^{1/2} \, , \tag{13}$$

see Červený (2001; eq. 5.3.5, where X_1 and X_2 are denoted P_2 and P_4). If $\beta_2 > \beta_1$, the term X_2 may become imaginary, specifically for $p > \beta_2^{-1}$ (overcritical incidence in the reference elastic medium). The term X_2 then reads:

$$X_2 = \mathrm{i} \bar{X}_2 = \mathrm{i}(\beta_2^2 p^2 - 1)^{1/2} \, . \tag{14}$$

Here, $\bar{X}_2$ is a real-valued term. The positive sign on the right-hand side of Eq. 14 results from the radiation condition applied in the reference elastic medium (decay of amplitudes of the transmitted wave with increasing distance from the interface). It corresponds to the above-mentioned use of the exponential factor $\exp[-\mathrm{i}\omega(t - p_k x_k)]$.

The term ξ in Eq. 12 can be determined from the specification of the incident, generally inhomogeneous, wave, see Eq. 7:

$$\xi = \frac{1}{2} Q_1^{-1} X_1 - \beta_1 D m_i N_i \, . \tag{15}$$

The term ξ may be positive or negative. It is zero for the attenuation vector $\mathbf{A}$ perpendicular to the normal $\mathbf{N}$ to the interface, see the third of Eq. 11.

The term $\xi^{(t)}$ can be determined by inserting the expressions for the vectors $P_i^{(t)}$ and $A_i^{(t)}$ given in Eq. 12 into the last of Eq. 5. From an inspection of Eq. 12, it is obvious that its separation into the propagation and attenuation vector is different for subcritical and overcritical incidence because the term X_2 behaves differently in subcritical and overcritical regions, see Eqs. 13 and 14. The same holds for $\xi^{(t)}$. Substitution of vectors $P_i^{(t)}$ and $A_i^{(t)}$ given in Eq. 12 with the real-valued $\xi^{(t)}$ into the last of Eq. 5 for overcritical incidence leads to unacceptable results. Thus the term $\xi^{(t)}$, which has the character of a directional cosine (similarly to X_2), must be considered imaginary:

$$\xi^{(t)} = \mathrm{i} \bar{\xi}^{(t)} \, . \tag{16}$$

In Eq. 16, $\bar{\xi}^{(t)}$ is a real-valued term.

Let us specify (12) for subcritical and overcritical incidence separately.

For subcritical incidence, Eq. 12 yields

$$P_i^{(t)} = P_i + \beta_1^{-1} X_1 N_i - \beta_2^{-1} X_2 N_i \; , \\ A_i^{(t)} = A_i + \beta_1^{-1} \xi N_i - \beta_2^{-1} \xi^{(t)} N_i \; . \tag{17}$$

The first two terms on the right-hand sides of expressions for $\mathbf{P}^{(t)}$ and $\mathbf{A}^{(t)}$ represent components of these vectors tangent to the interface, the last terms represent components perpendicular to it. The term X_2 in the first of Eq. 17 is given in Eq. 13. The term $\xi^{(t)}$ in the second of Eq. 17 can be determined from the substitution of the propagation and attenuation vectors from Eq. 17 into the last of Eq. 5. For $\xi^{(t)}$, we get

$$\xi^{(t)} = \frac{1}{2} \mathcal{Z} X_2^{-1} \; , \tag{18}$$

where

$$\mathcal{Z} = Q_2^{-1} - Q_1^{-1} \beta_2^2 p^2 - 2 X_1 \beta_1^{-1} \beta_2^2 D m_i N_i \; . \tag{19}$$

From Eq. 18, we can immediately see that $\xi^{(t)}$ is singular for critical incidence, for which $X_2 = 0$. Thus, we must expect that in the vicinity of the critical point of the reference elastic medium, $\xi^{(t)}$ will be determined inaccurately.

Since X_2 is always positive in the subcritical region, see Eq. 13, the propagation vector of the transmitted wave in Eq. 17 always points into the half-space where the transmitted wave propagates. Because the term $\xi^{(t)}$ controlling the vertical component of the attenuation vector may attain both positive or negative values, the attenuation vector of the transmitted wave may point into any of the two half-spaces. Its orientation into the upper half-space represents growth of the amplitude of the transmitted plane wave with distance from the interface, the phenomenon studied and explained by Richards (1984).

For overcritical incidence, taking into account Eqs. (14), (16) and (12) yields:

$$P_i^{(t)} = P_i + \beta_1^{-1} X_1 N_i + \beta_2^{-1} \bar{\xi}^{(t)} N_i , \\ A_i^{(t)} = A_i + \beta_1^{-1} \xi N_i - \beta_2^{-1} \bar{X}_2 N_i \; . \tag{20}$$

As in the case of subcritical incidence, the first two terms on the right-hand sides of expressions for $\mathbf{P}^{(t)}$ and $\mathbf{A}^{(t)}$ represent components of these vectors tangent to the interface, the last terms represent components perpendicular to it. The term $\bar{X}_2$ is given in Eq. 14. The term $\bar{\xi}^{(t)}$ can be again determined by inserting $\mathbf{P}^{(t)}$ and $\mathbf{A}^{(t)}$ from Eq. 20 into the last of Eq. 5. This leads to the expression for $\bar{\xi}^{(t)}$:

$$\bar{\xi}^{(t)} = -\frac{1}{2} \mathcal{Z} \bar{X}_2^{-1} \; , \tag{21}$$

where $\mathcal{Z}$ is given in Eq. 19. Equations 21, like 18, is inaccurate in the vicinity of critical incidence since $\bar{X}_2 \to 0$ when critical incidence is approached.

From Eq. 20, we can see that for overcritical incidence, the attenuation vector always points into the half-space, in which transmitted wave propagates since $\bar{X}_2$ is always positive, see Eq. 14. The vertical component of the propagation vector is, in this case, controlled by the perturbation term $\bar{\xi}^{(t)}$, which may be both positive or negative. This means that the propagation vector may point into any of the two half-spaces. The propagation vector pointing into the upper half-space, however, contradicts results obtained by Borcherdt (2009), according to which, the propagation vector of the transmitted wave should always point into the lower half-space. We return to this phenomenon later in the text.

We can see the important role of the term $\mathcal{Z}$, appearing in Eqs. 18 and 21, which it plays in the determination of vertical components of the slowness vector of the transmitted wave in an attenuating medium.

Let us first check behavior of the attenuation vector. In the subcritical region, the term $\mathcal{Z}$ controls the orientation of the attenuation vector of the transmitted wave, see Eq. 17. For $\mathcal{Z}$ positive, the attenuation vector points to the lower half-space, for $\mathcal{Z}$ negative, it points to the upper half-space. As mentioned above, Richards (1984) showed that the orientation of the attenuation vector to the upper half-space representing growth of the amplitude of the transmitted wave with the distance from the interface is an acceptable phenomenon. In the overcritical region, the attenuation vector always points into the lower half-space, see Eq. 20.

Let us now check the propagation vector. In the subcritical region, the propagation vector points always to the lower half-space, see Eq. 17, no matter the sign of the term $\mathcal{Z}$. The situation is different in the overcritical region. For $\mathcal{Z}$ positive, the propagation vector points to the lower half-space, for $\mathcal{Z}$ negative, it points to the upper half-space. In the latter case, we are in conflict with the observation of Borcherdt

(2009), according to which, the propagation vector of the transmitted wave must always point into the lower half-space. This result clearly indicates that there is a problem with the applicability of the correspondence principle.

Let us check under what conditions and where $\mathcal{Z}$ becomes negative. For a homogeneous wave incident from the upper half-space, the term $\mathcal{Z}$ in equation (19), reduces to $\mathcal{Z} = Q_2^{-1} - Q_1^{-1}\beta_2^2 p^2$. The term $\mathcal{Z}$ is positive for normal incidence ($p = 0$) and it becomes zero when $p^2 = \beta_1^{-2}\sin^2 i = \beta_2^{-2} Q_1 Q_2^{-1}$, i.e., when

$$\sin^2 i = Q_1 Q_2^{-1}\beta_1^2\beta_2^{-2} = Q_1 Q_2^{-1}\sin^2 i^* . \qquad (22)$$

Here, i^* denotes the critical angle. From Eq. 22 we can see that for $Q_1 = Q_2$, i.e., when the attenuation is the same in both half-spaces, $\mathcal{Z}$ becomes zero exactly for $i = i^*$, i.e. for critical incidence. For $Q_2 > Q_1$ (realistic situation), changing $\mathcal{Z}$ from positive to negative occurs in the subcritical region, for $Q_2 < Q_1$ (rare case) in the overcritical region.

If the incident wave is inhomogeneous, the last term in Eq. 19 causes a shift of the change of the sign of $\mathcal{Z}$. If the last term in Eq. 19 is negative, the change of the sign of $\mathcal{Z}$ shifts towards larger values of the incidence angle i, i.e., towards the overcritical region. If the last term in Eq. 19 is positive, the change of the sign of $\mathcal{Z}$ shifts towards the subcritical region. Except for normal incidence, the scalar product $\mathbf{m} \cdot \mathbf{N}$ in Eq. 19 is negative. Therefore, if D (or γ) is positive, the change of the sign of $\mathcal{Z}$ shifts towards the overcritical region. If D (or γ) is negative, the change shifts towards the subcritical region.

It follows from the above analysis that when $Q_2 < Q_1$, the propagation vector of the transmitted wave generated by an incident homogeneous wave starts to point to the upper half-space for an overcritical incidence. When $Q_2 > Q_1$, the propagation vector behaves like this since critical incidence. As mentioned above, this contradicts results of Borcherdt (2009), and indicates inapplicability of the correspondence principle. In Section 5, we show that the described behavior of the propagation vector of the transmitted wave leads to distortions in the reflection and transmission coefficients, which may be artificially corrected.

4 Reflection and transmission coefficients

Using the results of previous sections, coefficients in Eq. 1 can be rewritten in the following form:

$$R = \frac{A-B}{A+B} , \qquad T = \frac{2A}{A+B} , \qquad (23)$$

where

$$\begin{aligned} A &= \rho_1\beta_1^2(1 - \mathrm{i}Q_1^{-1})(P_i + \mathrm{i}A_i)N_i , \\ B &= \rho_2\beta_2^2(1 - \mathrm{i}Q_2^{-1})(P_i^{(t)} + \mathrm{i}A_i^{(t)})N_i . \end{aligned} \qquad (24)$$

The propagation and attenuation vectors, $\mathbf{P}$ and $\mathbf{A}$, related to the incident wave, are given in Eq. 7. The propagation and attenuation vectors, $\mathbf{P}^{(t)}$ and $\mathbf{A}^{(t)}$, related to the transmitted wave, are given in Eq. 17 for the subcritical and in Eq. 20 for overcritical incidence.

If attenuation is the same on both sides of the interface ($Q_1 = Q_2 = Q$) and the incident wave is homogeneous, i.e., $\xi = \frac{1}{2}Q^{-1}X_1$, then $\xi^{(t)} = \frac{1}{2}Q^{-1}X_2$ and $\bar{\xi}^{(t)} = \frac{1}{2}Q^{-1}\bar{X}_2$. As a consequence, Eqs. 23 and 24 reduce to Eq. 1, in which the slowness vectors $\mathbf{p}$ and $\mathbf{p}^{(t)}$ are replaced by the propagation vectors $\mathbf{P}$ and $\mathbf{P}^{(t)}$, respectively. This confirms Buchen's (1971) statement that "when Q is the same for both media, the waves display properties similar to the case of perfect elasticity". See also Krebes (1983).

If velocities and densities are the same in both half-spaces, $\beta_2 = \beta_1$, $\rho_2 = \rho_1$, but $Q_2 \neq Q_1$, the reflection coefficient is non-zero. This case has been studied by Lines et al. (2008).

If media on both sides of the interface are elastic and the incident wave is homogeneous, Eq. 23 reduces to Eq. 1 for elastic media. In this case, Q_1^{-1} and Q_2^{-1} are zero and due to the homogeneity of the incident wave, ξ, $\xi^{(t)}$ and $\bar{\xi}^{(t)}$ are also zero.

Coefficients in Eq. 23 can be rewritten into the form of sums of coefficients for reference elastic media plus corrections related to the weak attenuation, see Buchen (1971, eq.8.1). This process is, however, connected with a certain loss of accuracy caused by neglecting higher-order terms. The simplified reflection and transmission coefficients can be expressed as:

$$R = R_{el} + \Delta C , \qquad T = T_{el} + \Delta C . \qquad (25)$$

In Eq. 25, R_{el} and T_{el} represent the reflection and transmission coefficients in the reference elastic medium, and ΔC is the correction due to weak attenuation. It is the same for both the reflection and transmission coefficients.

The reflection and transmission coefficients R_{el} and T_{el} are equivalent to Eq. 1, and are:

$$R_{el} = \frac{\rho_1\beta_1X_1 - \rho_2\beta_2X_2}{\rho_1\beta_1X_1 + \rho_2\beta_2X_2}\,, \qquad T_{el} = \frac{2\rho_1\beta_1X_1}{\rho_1\beta_1X_1 + \rho_2\beta_2X_2}\,. \tag{26}$$

The correction term ΔC reads:

$$\Delta C = 2\mathrm{i}\rho_1\beta_1\rho_2\beta_2\frac{[X_2\xi - X_1\xi^{(t)} + X_1X_2(Q_2^{-1} - Q_1^{-1})]}{(\rho_1\beta_1X_1 + \rho_2\beta_2X_2)^2}\,. \tag{27}$$

The terms X_1 and ξ are given in Eqs. 13 and 15, respectively. For subcritical incidence, X_2 and $\xi^{(t)}$ are given in Eqs. 13 and 18, respectively, for overcritical incidence, $X_2 = \mathrm{i}\bar{X}_2$ and $\xi^{(t)} = \mathrm{i}\bar{\xi}^{(t)}$, see Eqs. 14 and 16 with 21, respectively.

If the attenuation is the same on both sides of the interface ($Q_1 = Q_2 = Q$) and the incident wave is homogeneous, the numerator in Eq. 27 vanishes resulting in $\Delta C = 0$. The coefficients R and T in Eq. 25 reduce to coefficients corresponding to the elastic reference medium.

The term ΔC also vanishes when both half-spaces are elastic and the incident wave is homogeneous, i.e., when $Q_1^{-1} = Q_2^{-1} = \xi = \xi^{(t)} = 0$.

5 Numerical tests

In this section, we present examples of the use of formulae (23) with (24). We concentrate on models with higher S-wave velocities in the lower half-space, because in the elastic case, they display the critical incidence phenomenon. Treatment of models with opposite ratios of S-wave velocities presents no problem. For testing purposes, we use models M1, M2, and M3 proposed by Brokešová and Červený (1998).

We first concentrate on models M2 and M3 because they are characterized by attenuation, which might be considered moderate. Both models have the same S-wave velocities and densities, $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s and $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, respectively. The models M2 and M3 differ by the

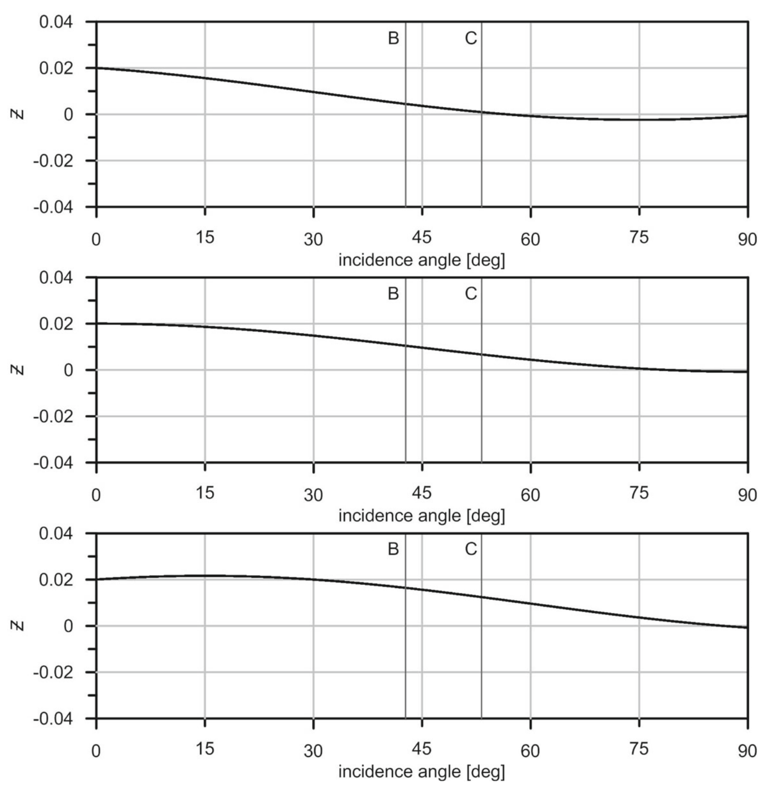


Fig. 1 The term $\mathcal{Z}$, see Eq. 19, as a function of the incidence angle for incident homogeneous wave (middle) and inhomogeneous waves with $\gamma = -30°$ (top) and $\gamma = 30°$ (bottom) for model M3 of Brokešová and Červený (1998) ($Q_1 > Q_2$): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 75$, $Q_2 = 50$. Vertical lines B and C indicate positions of the Brewster and critical angles in the reference elastic medium, respectively

values of the quality factors Q. In the more realistic ($Q_1 < Q_2$) model M2, $Q_1 = 50$ and $Q_2 = 75$, in the model M3, the quality factors are interchanged, $Q_1 = 75$ and $Q_2 = 50$. Greater velocities in the lower half-space than in the upper half-space result in the existence of the critical angle in the reference elastic medium, at approximately 52°. Another important phenomenon is the existence of the Brewster angle at approximately 43°, at which the modulus of the reflection coefficient in the reference elastic medium vanishes (the term Brewster angle is taken from electromagnetics, see, e.g., Born and Wolf 1959). The Brewster angle is caused by the negative value of the reflection coefficient (26) at the normal incidence and its positive value (equal to 1) at the critical point.

The attenuation in the model close to the model M1 of Brokešová and Červený (1998) cannot be considered weak. The quality factors are $Q_1 = 15$, $Q_2 = 22$, and velocities and densities are $\beta_1 = 1.44$ and $\beta_2 = 2.08$ km/s, $\rho_1 = 2.0$ and $\rho_2 = 2.0$ kg/m^3. We use the model M1 to illustrate how WAC performs in media with stronger attenuation. The critical angle in the reference elastic model for the model M1 is at approximately 44°, and the Brewster angle is at approximately 35°.

Figures 1 and 2 illustrate the behavior of the term $\mathcal{Z}$, see Eq. 19, in models M3 and M2, respectively. Both figures show the variation of the term $\mathcal{Z}$ with the angle of incidence i of the incident inhomogeneous waves with attenuation angles $\gamma = -30°$ (top) and $\gamma = 30°$ (bottom), and of the incident homogeneous wave ($\gamma = 0°$) in the middle. In Fig. 1, the term $\mathcal{Z}$ in the model M3 ($Q_1 > Q_2$) is positive for all considered attenuation angles in the subcritical region and around critical incidence (for $\gamma = -30°$ only slightly). This means that the propagation vector of the transmitted wave points into the lower half-space in subcritical region and even around critical incidence. The term $\mathcal{Z}$ becomes slightly negative in the overcritical region, which results in the unacceptable orientation of the propagation vector of the transmitted wave into the upper half-space. In Fig. 2, the behavior is shown of the term $\mathcal{Z}$ in the model M2, the situation is different.

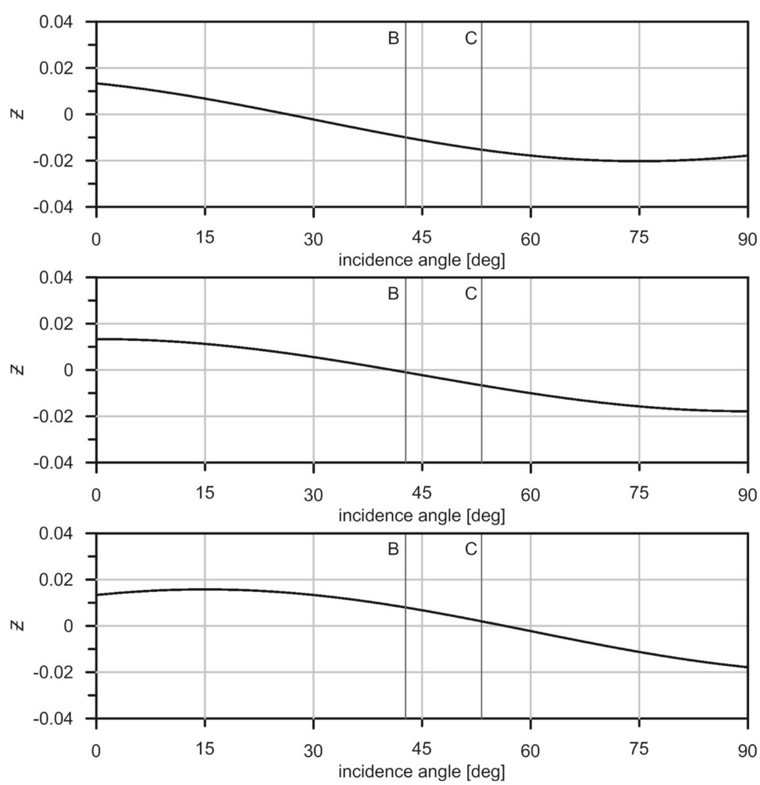


Fig. 2 The term $\mathcal{Z}$, see Eq. 19, as a function of the incidence angle for incident homogeneous wave (middle) and inhomogeneous waves with $\gamma = -30°$ (top) and $\gamma = 30°$ (bottom) for model M2 of Brokešová and Červený (1998) ($Q_1 < Q_2$): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$. Vertical lines B and C indicate positions of the Brewster and critical angles in the reference elastic medium, respectively

In this case, the term $\mathcal{Z}$ becomes negative even before critical incidence of the homogeneous wave and for the inhomogeneous incident wave with $\gamma = -30°$. This leads to a discontinuous change of the orientation of the propagation vector in the critical region. As a result, the propagation vector of the transmitted wave points to the upper half-space in the whole subcritical region. The only exception is the case of the incidence of the inhomogeneous wave with $\gamma = 30°$, for which the term $\mathcal{Z}$ is positive within the whole subcritical region as in Fig. 1.

Effects of the above described behavior of the term $\mathcal{Z}$ can be observed in Figs. 3 and 4. These figures show the orientation and relative sizes of the propagation and attenuation vectors of the transmitted wave generated by the incidence of a homogeneous and two inhomogeneous waves, again in models M3 and M2. Red is used to indicate the propagation vector, green the attenuation vector. The vectors are determined from Eqs. 17 and 20. The three attenuation angles of incident waves are again $\gamma = -30°$ (top), $\gamma = 0°$ (middle) and $\gamma = 30°$ (bottom). The incidence angles i vary from 0° to 80°, with the step of 10°. The attenuation vector is very small in the subcritical region. Therefore, it is amplified in the subcritical region by a factor of 100 in Figs. 3 and 4.

In Fig. 3, we can see the results for the model M3. This is a model, which Krebes and Daley (2007) characterize as a model without phase discrepancy. This model, with $Q_1 > Q_2$ is, however, not very realistic because usually higher impedance implies higher Q. We can see that the orientation of the propagation and attenuation vectors vary continuously within the critical region. The attenuation vector points into the lower half-space. In the overcritical region, it slightly deviates from the vertical due to either the attenuation in

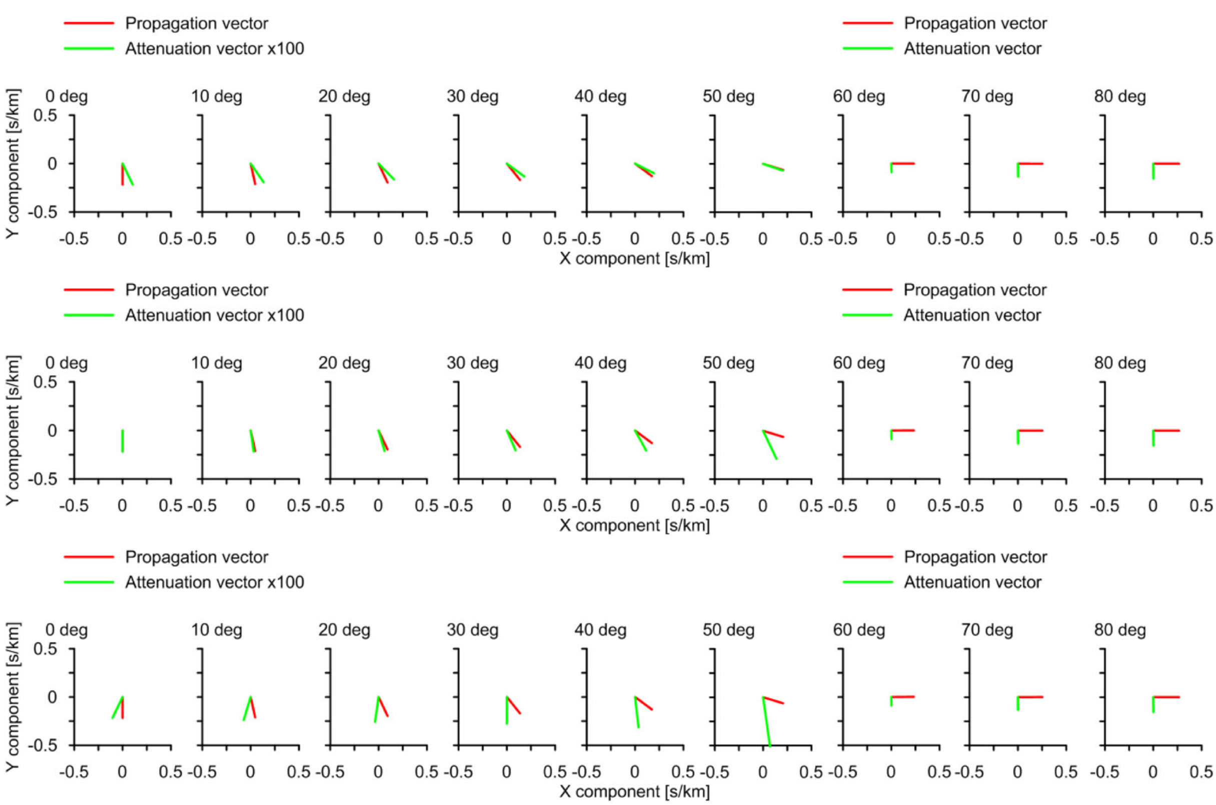


Fig. 3 Propagation (red) and attenuation (green; amplified in the subcritical region) vectors of the transmitted wave generated by the incident homogeneous (middle) and inhomogeneous (top $\gamma = -30°$ and bottom $\gamma = 30°$) waves in the model M3 of Brokešová and Červený (1998) ($Q_1 > Q_2$): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 75$, $Q_2 = 50$

the upper half-space and/or inhomogeneity of the incident wave. The propagation vector also points mostly into the lower half-space although only slightly in the overcritical region. The exception is the propagation vector of the transmitted wave corresponding to the incident inhomogeneous wave with $\gamma = -30°$. The propagation vector points slightly into the upper half-space for angles of incidence $i \geq 60°$, as indicated in the top frame of Fig. 1. Its vertical component represents, however, only a small perturbation, and thus it is invisible in Fig. 3. Although small, the deviation of the propagation vector into the upper half-space is unacceptable. It contradicts the results of Borcherdt (2009), according to which, it should always point into the lower half-space. The middle frame of Fig. 3 confirms the well-known fact (see, e.g., Červený 2007; Borcherdt 2009) that a homogeneous wave incident at an interface separating two attenuating half-spaces generates, generally, inhomogeneous waves. In the top frame, we can see that, on the contrary, the incident inhomogeneous wave may generate a homogeneous wave.

In Fig. 4, results for the more realistic model M2 with $Q_1 < Q_2$ are presented in the same way as in Fig. 3. For the negative attenuation angle (the top frame) of the incident wave and even for some angles of incidence of a homogeneous wave, we can see that the attenuation vector of the transmitted wave in the subcritical region points into the upper half-space, which is acceptable as explained by Richards (1984). In the critical region, the orientation of the attenuation vector changes abruptly. The propagation vector of the transmitted wave points into the lower half-space in the subcritical region. However, for $\gamma = -30°$ and $0°$, immediately behind critical incidence, the propagation vector attains an unacceptable orientation into the

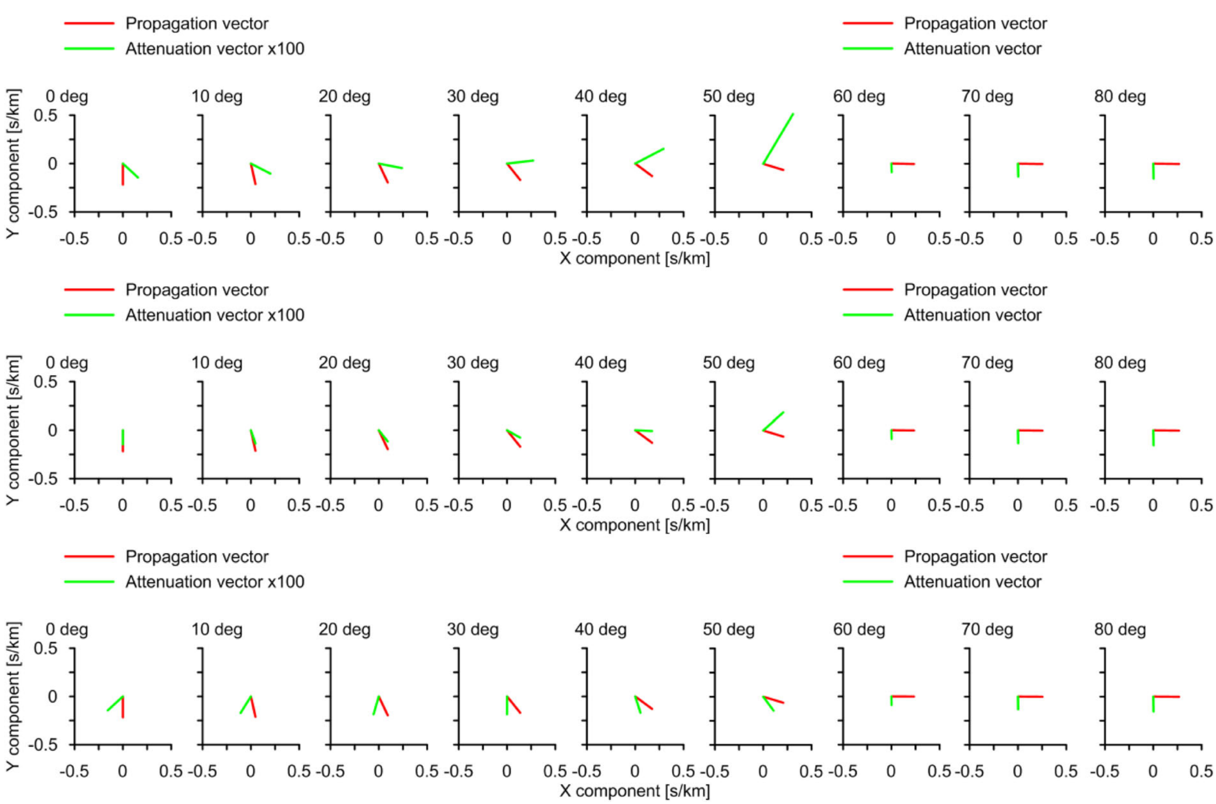


Fig. 4 Propagation (red) and attenuation (green; amplified in the subcritical region) vectors of the transmitted wave generated by the incident homogeneous (middle) and inhomogeneous (top $\gamma = -30°$ and bottom $\gamma = 30°$) waves in the model M2 of Brokešová and Červený (1998) ($Q_1 < Q_2$): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$)

upper half-space (Borcherdt 2009). Its deviation from the interface is, however, small and thus effectively invisible.

In the following figures, we concentrate on SH reflection and transmission coefficients in models M3, M2, and M1 for incident homogeneous and inhomogeneous waves. To estimate the accuracy of WAC coefficients for incident homogeneous waves, in Figs. 5, 6, 7, 8, 9, and 12, we compare them with coefficients calculated from the formulae of Daley and Krebes (2015), obtained without the use of the correspondence principle. We also show coefficients calculated for the homogeneous wave incident at the interface between reference elastic half-spaces. This allows us to estimate the effects of the attenuation.

Modulus (top frame) and phase (bottom frame) of the SH reflection coefficient corresponding to the incident homogeneous wave in the model M3 is shown in Fig. 5. Curves obtained by the first formula of Eqs. 23 with 24 are red dotted. Green dashed curves correspond to the modulus and phase calculated from formulae of Daley and Krebes (2015). As a reference, we also show the modulus and phase corresponding to the reference elastic medium (black solid curve). We can see a nearly perfect coincidence of WAC and Daley and Krebes (2015) curves for all angles of incidence. The only slight difference can be observed in the phase frame in the vicinity of the Brewster angle and of the critical angle, where the WAC approximation is inapplicable. As observed previously (Brokešová and Červený 1998), attenuation smoothes the curves corresponding to the elastic case. This smoothing, although hardly visible in the modulus frame, also includes the vicinity of the Brewster angle. Due to this smoothing, the modulus of the reflection coefficient in the anelastic medium is non-zero at the Brewster angle, although only negligibly. By comparing red and green curves with the black curve corresponding to the reference

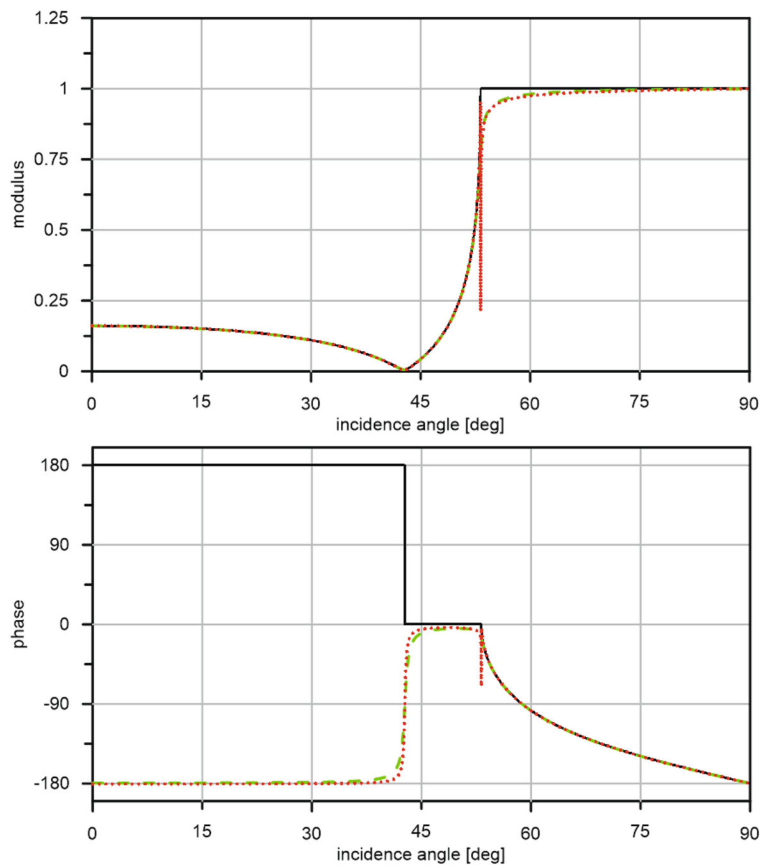


Fig. 5 Comparison of moduli (top) and phases (bottom) of the SH plane-wave reflection coefficient in the model M3 ($Q_1 > Q_2$) of Brokešová and Červený (1998): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 75$, $Q_2 = 50$. Elastic reference (solid black), Daley and Krebes (2015) (dashed green), formulae (23) and (24) of this paper (dotted red). Incident homogeneous wave

elastic case, we can see that except in the vicinity of the Brewster and critical angles, the effect of attenuation on the reflection coefficient is very small.

In Fig. 6, the same results as in Fig. 5 are shown, but for the model M2 ($Q_1 < Q_2$). We can see a perfect coincidence of the WAC (red dotted) and Daley and Krebes (2015) (green dashed) curves in the subcritical region. In the overcritical region, we can, however, observe significant differences in the modulus (WAC modulus larger than 1), which decrease with the increasing angle of incidence. It is obviously the consequence of the abrupt change of the orientation of the propagation and also attenuation vector for angles of incidence varying from the subcritical to the overcritical incidence, see Fig. 4, and also of the unacceptable orientation of the propagation vector of the transmitted wave into the upper half-space. As it was shown, it is due to the negative values of the term $\mathcal{Z}$, see Eq. 19, starting in the subcritical region, see Fig. 2. One can speculate what would happen if the sign of the term $\mathcal{Z}$ is changed artificially once it turns negative. This would cause the propagation and attenuation vectors to vary smoothly through the critical region, keeping the propagation vector to point into the lower half-space. The result of such a manipulation is shown in Fig. 7. We can see a nearly perfect fit of red dotted and green dashed curves for all angles of incidence. We thus have a tool for making the approximation introduced in this paper applicable even in the problematic, but frequent, situation of negative $\mathcal{Z}$ in the subcritical region. We should, however, keep in mind what is the price of this achievement. The artificial change of the sign of $\mathcal{Z}$ represents the violation of the constraint relation (4) and of the generalized Snell's law (8), from which Eqs. 17 and 20 were derived. It is again an indication that the correspondence principle is not applicable in the overcritical region. Let us add that the artificial change of the sign of the term $\mathcal{Z}$, which became negative in the overcritical region (for example, for the incident homogeneous wave in

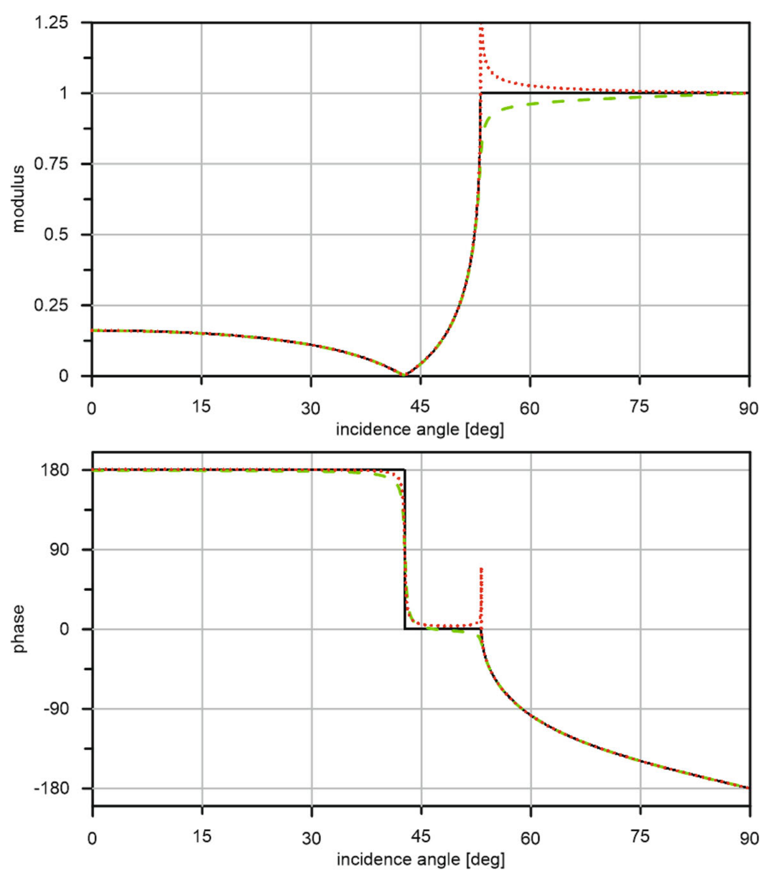


Fig. 6 Comparison of moduli (top) and phases (bottom) of the SH plane-wave reflection coefficient in the model M2 ($Q_1 < Q_2$) of Brokešová and Červený (1998): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$. Elastic reference (solid black), Daley and Krebes (2015) (dashed green), formulae (23) and (24) of this paper (dotted red). Incident homogeneous wave

model M3, see Fig. 1) would lead to the distortions in the modulus. Only the artificial change of the sign of the term $\mathcal{Z}$ when it becomes negative in the subcritical region leads to acceptable results.

In Fig. 8, we show, also for the more realistic model M2, the modulus (top) and phase (bottom) of the transmission coefficient. Except for the critical region, the fit of results obtained from Eq. 23 and green dashed curves generated by the code of Daley and Krebes (2015) is very good. The trick with the change of the sign of $\mathcal{Z}$ in the subcritical region leads, however, to an even better fit, see Fig. 9. As in the case of reflection, we can see that the difference of the transmission coefficients calculated in the attenuative model (red and green curves) and of the coefficient calculated in the reference elastic model (black) is, except within the narrow vicinity of critical incidence, negligible.

The effect of inhomogeneity of the incident wave on the reflection coefficient is presented in Figs. 10 and 11 for the model M2. Moduli (top) and phases (bottom) of SH reflection coefficients for positive attenuation angles γ (the attenuation vector rotated clockwise from the propagation vector) of incident waves are shown in Fig. 10, and for negative angles γ in Fig. 11. WAC results (red), in which the sign of $\mathcal{Z}$ is changed if it becomes negative in the subcritical region, are compared with the results for the reference elastic case (black). In Fig. 10, attenuation angles of incident waves 15° (dots), 30° (short dashes), and 45° (long dashes) are considered. Reflection coefficients corresponding to incident inhomogeneous waves with negative attenuation angles −15° (dots), −30° (short dashes), and −45° (long dashes) are shown in Fig. 11. Coefficient for the reference elastic case is shown again in black. With varying values of the attenuation

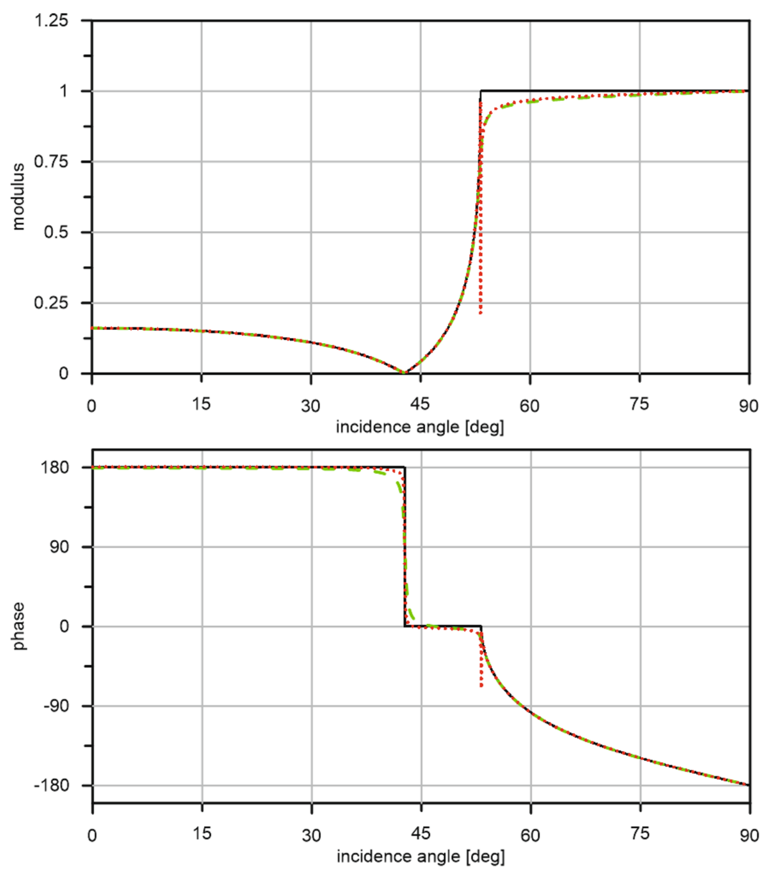


Fig. 7 Comparison of moduli (top) and phases (bottom) of the SH plane-wave reflection coefficient in the model M2 ($Q_1 < Q_2$) of Brokešová and Červený (1998): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$. Elastic reference (solid black), Daley and Krebes (2015) (dashed green), formulae (23) and (24) of this paper with change of sign of $\mathcal{Z}$ when it becomes negative (dotted red). Incident homogeneous wave

angle of the incident wave, the form of reflection coefficients varies, most significantly close to the Brewster and critical angles. It is of interest to point out that closest to the form of the reflection coefficient for the elastic case is not the coefficient corresponding to the incident homogeneous wave, but the coefficient corresponding to the inhomogeneous wave with $\gamma \sim 26°$. We can see that weak inhomogeneity of the incident wave leads to observable differences from the reflection coefficient for the reference elastic case only in the vicinity of the Brewster and critical angles.

In Fig. 12, we use a slightly modified model M1 of Brokešová and Červený (1998) whose attenuation cannot be considered weak (Q_1=15, Q_2=22). We compare again the WAC results (red dotted) with results obtained from Daley and Krebes (2015) formula (green dashed). Moduli in the subcritical region and phases in the overcritical region fit perfectly. Comparison of moduli in the overcritical and of phases in the subcritical regions show some differences. They, however, only slightly exceed the differences observable in Fig. 7, generated for a considerably weaker attenuation. The difference of the coefficient calculated for the medium with stronger attenuation and of the reference coefficient calculated for the elastic case (black solid) extends farther into the overcritical region.

6 Discussion

We used the weak-attenuation concept (WAC), within which attenuation is considered to be a small perturbation of the reference elastic state, and we applied it to the approximate evaluation of SH-wave reflection and

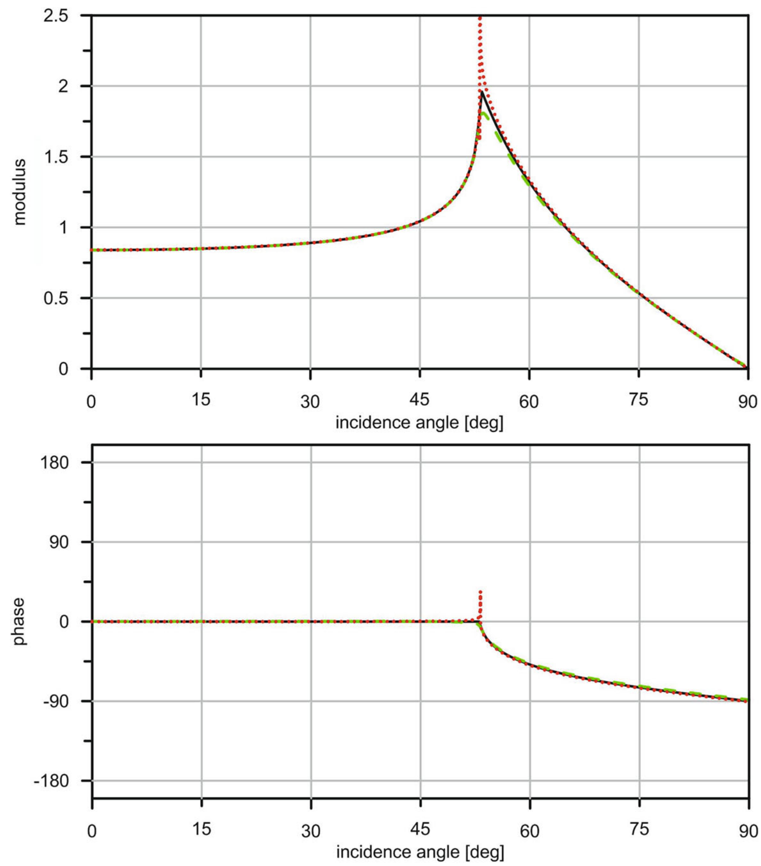
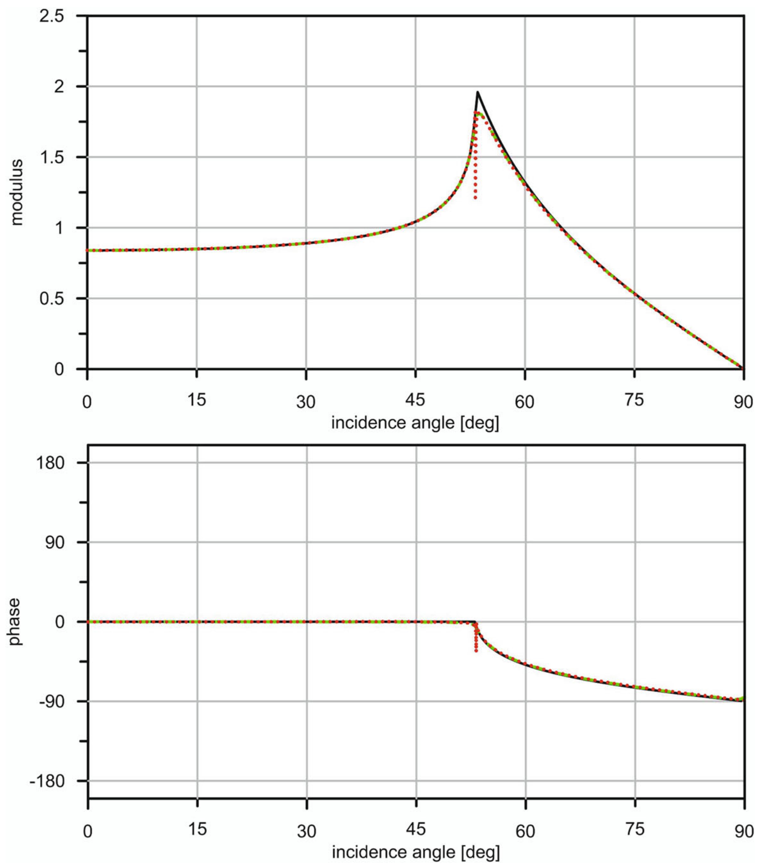


Fig. 8 Comparison of moduli (top) and phases (bottom) of the SH plane-wave transmission coefficient in the model M2 ($Q_1 < Q_2$) of Brokešová and Červený (1998): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$. Elastic reference (solid black), Daley and Krebes (2015) (dashed green), formulae (23) and (24) of this paper (dotted red). Incident homogeneous wave

transmission coefficients at an interface separating two homogeneous isotropic attenuating half-spaces. In the derivations of formulae for reflection and transmission coefficients, we used the correspondence principle (wave propagation results in attenuating media can be obtained as the results in elastic media by replacing real-valued elastic parameters by their complex-valued anelastic counterparts; Bland 1960). In the past, the correspondence principle was used successfully for an approximate evaluation of effects of attenuation on wave propagation inside layers (Moczo et al. 1987; Gajewski and Pšenčík 1992) and it was applied it in ray-based program packages (Červený and Pšenčík 1984; Gajewski and Pšenčík 1990). It was of interest to study effects of weak attenuation on the reflection and transmission process. Use of the correspondence principle for the derivation of approximate reflection and transmission coefficients in attenuating media was the natural choice. We used it despite the fact that Borcherdt (2009, 2020) pointed out that the conditions of applicability of the correspondence principle are not satisfied in the overcritical region. Indeed, we found that its use leads to violations of some of the constraint relations, which the waves propagating in attenuative media must satisfy. Nevertheless, we arrived at two useful observations. First, we found that if attenuation is weak and incident waves are homogeneous or weakly inhomogeneous, effects of attenuation on reflection and transmission coefficients outside vicinities of critical incidence or the Brewster angle are negligible. Therefore, by ignoring the effects of attenuation on the reflection and transmission process, Gajewski and Pšenčík (1992) did not greatly affect the accuracy of their computations. Second, if we wish to consider the effects of attenuation on the reflection and transmission, we found that

Fig. 9 Comparison of moduli (top) and phases (bottom) of the SH plane-wave transmission coefficient in the model M2 ($Q_1 < Q_2$) of Brokešová and Červený (1998): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$. Elastic reference (solid black), Daley and Krebes (2015) (dashed green), formulae (23) and (24) of this paper with change of sign of $\mathcal{Z}$ when it becomes negative (dotted red). Incident homogeneous wave



an artificial modification of the slowness vector of the transmitted wave leads to useful results of acceptable accuracy, which can be used in the above-mentioned codes.

The basic role in the study is played by slowness vectors of incident and transmitted waves. Complex-valued slowness vectors of the incident and transmitted waves satisfy (i) corresponding approximate constraint relations resulting from the equation of motion, (ii) Snell's law, and (iii) the radiation condition. With the above conditions satisfied, the formulae for the reflection and transmission coefficients have the following properties.

The reflection and transmission coefficients satisfy naturally the "continuity" criterion, which means that with decreasing attenuation the formulae converge to the reflection and transmission coefficients of waves reflected and transmitted at an interface separating two elastic media.

The accuracy of the approximate formulae for the reflection and transmission coefficients is high in the subcritical region. Acceptable accuracy in this region is obtained even for Q's $\sim$ 20. The formulae are singular for critical incidence in the reference elastic medium. Therefore, their accuracy is low in this vicinity. The extent of this vicinity increases with decreasing values of Q, increasing contrast of Q^{-1} across the interface, and varies more significantly, with varying inhomogeneity of the incident wave. Another region of decreased accuracy for lower values of Q is in the vicinity of the Brewster angle, see Fig. 12.

The presented results show that the effects of attenuation on the reflection and transmission coefficients are rather weak (weaker than the effect of attenuation

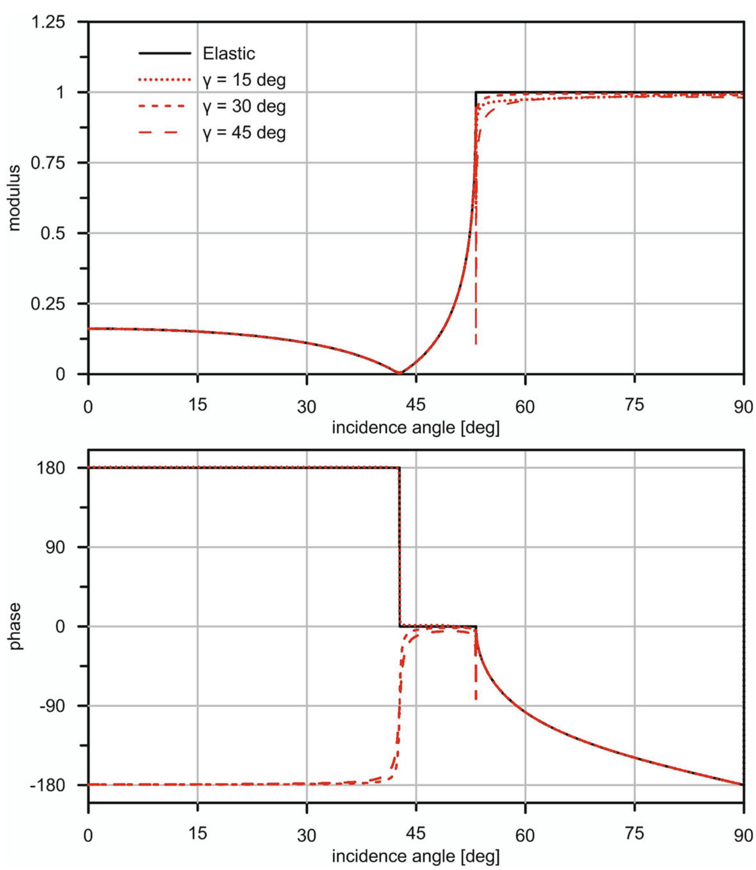


Fig. 10 Comparison of moduli (top) and phases (bottom) of the SH plane-wave reflection coefficient in the model M2 ($Q_1 < Q_2$) of Brokešová and Červený (1998): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$ for varying inhomogeneity of the incident wave: $\gamma = 15°$, $\gamma = 30°$ and $\gamma = 45°$. Formulae (23) and (24) red, elastic reference solid black

inside layers), particularly for subcritical incidence. Thus, it seems that ignoring the effects of attenuation on reflection and transmission in the computations of the whole wavefield does not affect the results much. If the consideration of effects of attenuation on reflection and transmission coefficients is required, the WAC formulae for the coefficients can be used without problems for subcritical incidence. Their use can be extended to the overcritical region by changing the sign of the term $\mathcal{Z}$ introduced in the text when it becomes negative in the subcritical region. This step corresponds to the enforced change of the incorrect orientation of the propagation vector of the transmitted wave in the overcritical region. With the correct orientation of the propagation vector, reflection and transmission coefficients fit the coefficients calculated without the use of the correspondence principle (Daley and Krebes 2015). As described in the text, the enforced change of the orientation of the propagation vector leads, however, to the violation of some of the constraint relations. This is a price we pay for the use of the correspondence principle in the reflection and transmission problem in the overcritical region.

A similar procedure applied to SH waves in this paper could be extended to other types of waves propagating in isotropic and also anisotropic, attenuating media. An important step before the application to anisotropic media is to use the WAC formulae in the calculation of SH-wave ray synthetic seismograms in layered, isotropic, attenuating media, and their comparison with other independent wave-modeling methods. A promising candidate is the method used by Ursin et al. (2017). Later, the study can be extended to the calculation of P- and S-wave ray synthetic seismograms, in which slowness vectors of the studied incident waves are not confined to the plane of incidence. Again, a comparison of WAC seismograms with seismograms generated by other independent wave-modeling methods is desirable.

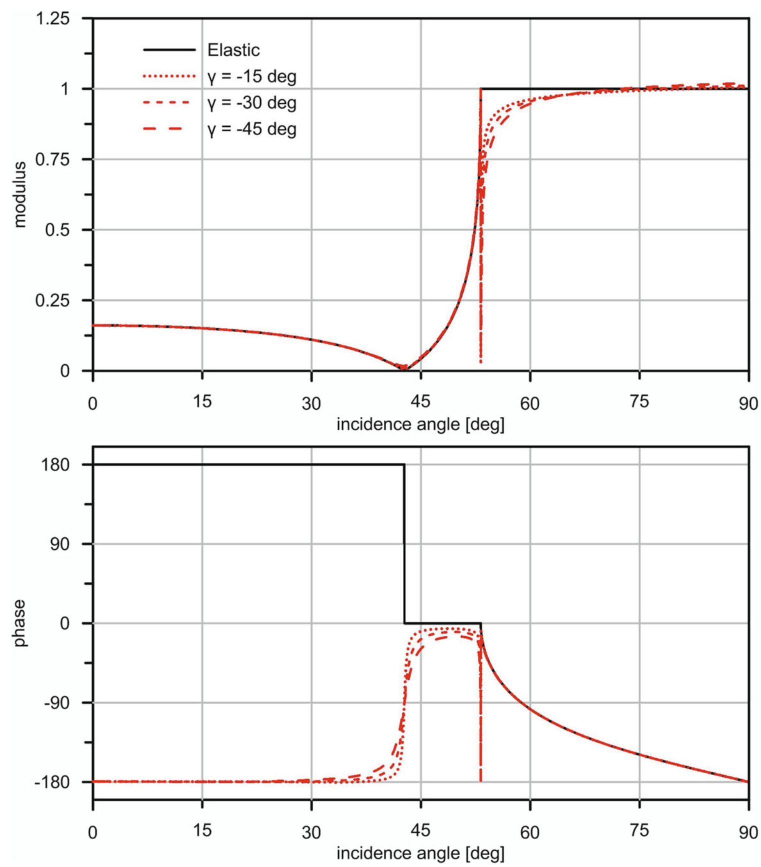
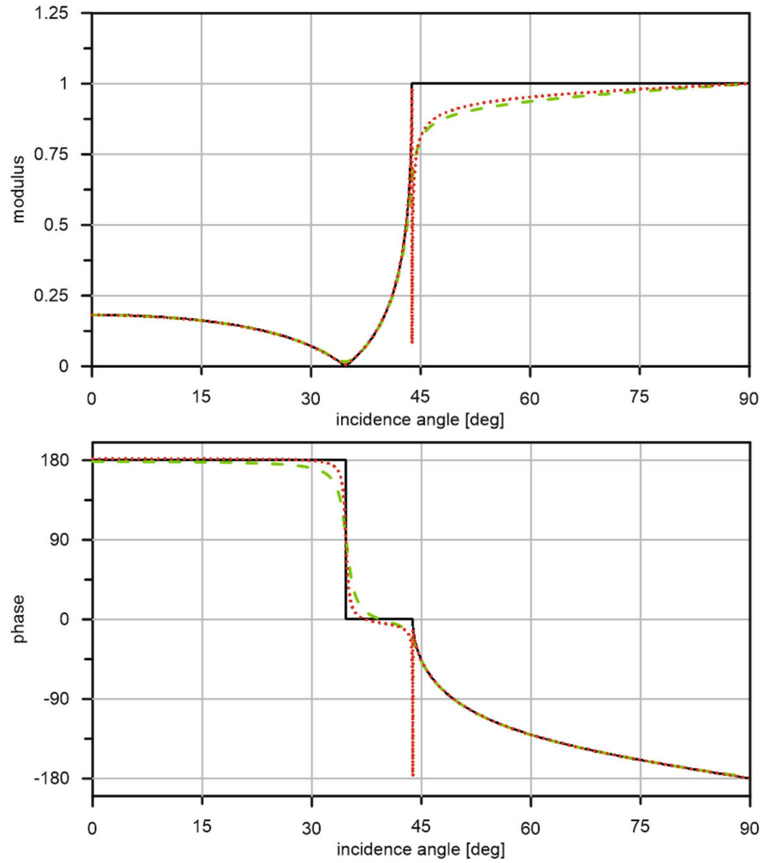


Fig. 11 Comparison of moduli (top) and phases (bottom) of the SH plane-wave reflection coefficient in the model M2 ($Q_1 < Q_2$) of Brokešová and Červený (1998): $\beta_1 = 3.698$, $\beta_2 = 4.618$ km/s, $\rho_1 = 2.98$, $\rho_2 = 3.3$ kg/m^3, $Q_1 = 50$, $Q_2 = 75$ for varying inhomogeneity of the incident wave: $\gamma = -15°$, $\gamma = -30°$ and $\gamma = -45°$. Formulae (23) and (24) red, elastic reference solid black

Fig. 12 Comparison of moduli (top) and phases (bottom) of the SH plane-wave reflection coefficient in the model close M1 ($Q_1 < Q_2$) of Brokešová and Červený (1998): $\beta_1 = 1.44$, $\beta_2 = 2.08$ km/s, $\rho_1 = 2.0$, $\rho_2 = 2.0$ kg/m^3, $Q_1 = 15$, $Q_2 = 22$. Elastic reference (solid black), Daley and Krebes (2015) (dashed green), formulae (23) and (24) of this paper with change of sign of $\mathcal{Z}$ when it becomes negative (dotted red). Incident homogeneous wave



7 Conclusions

We derived approximate expressions for the SH-wave reflection and transmission coefficients at interfaces separating two homogeneous isotropic, attenuating half-spaces. They can be used in ray-based codes to extend their applicability to layered media. The expressions for coefficients are inaccurate in the vicinity of critical incidence in a reference elastic medium. Outside critical region their accuracy seems to be satisfactory. The presented tests indicate that attenuation affects waves propagating inside layers more significantly than their reflection or transmission processes. Thus, ignoring effects of attenuation on the reflection and transmission process may not lead to significant loss of accuracy. Corresponding tests of complete seismic wavefields in layered attenuating media are under preparation.

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Code availability At the authors

Compliance with Ethical Standards

Competing interests The authors declare no competing interests.

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