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Variance asymptotics for the area of planar cylinder processes generated by Brillinger-mixing point processes

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Abstract. We introduce cylinder processes in the plane defined as union sets of dilated straight lines (appearing as mutually overlapping infinitely long strips) generated by a stationary independently marked point process on the real line, where the marks describe the width and orientation of the individual cylinders. We study the behavior of the total area of the union of strips contained in a space-filling window ϱK as $\varrho \to \infty$. In the case the unmarked point process is Brillinger mixing, we prove the mean-square convergence of the area fraction of the cylinder process in ϱK. Under stronger versions of Brillinger mixing, we obtain the exact variance asymptotics of the area of the cylinder process in ϱK as $\varrho \to \infty$. Due to the long-range dependence of the cylinder process, this variance increases asymptotically proportionally to ϱ^3.

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1 Introduction and preliminaries

Cylinder processes (CPs) in $\mathbb{R}^d$ defined as countable unions of dilated affine subspaces $\mathbb{R}^k$, $k = 1, \ldots, d-1$, are basic random set models in stochastic geometry; see, for example, [15, 19, 21] and [16]. Meanwhile CPs also receive much attention due to numerous applications (for $d = 2, 3$) in modern technologies as models for materials consisting of long fibers or to model dynamic telecommunication networks. Until now, so far as we know, asymptotic properties of CPs in expanding domains were exclusively studied under Poisson assumptions; see [13, 14, 20] and [1]. The Poisson property of the generating stationary point process (PP) provides, among others, the stationarity of the associated CP. In this paper, we focus on planar CPs generated by a more general class of stationary independently marked PPs on $\mathbb{R}^1$. We assume that the corresponding

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unmarked (ground) PP is Brillinger mixing, that is, its reduced cumulant measures of any order exist and have finite total variation.

Throughout this paper, all random elements are defined on a common probability space $[\Omega, \mathcal{F}, \mathbf{P}]$, and by $\mathbf{E}$ and Var we denote the expectation and variance with respect to $\mathbf{P}$. Next, we describe a CP in $\mathbb{R}^2$ in terms of its generating stationary independently marked PP on $\mathbb{R}^1$. Let (Φ_0, R_0) be the generic random vector taking values in the mark space $[0,\pi) \times [0,\infty)$ that describes the orientation Φ_0 and cross-section (or base) $\Xi_0 := [-R_0, R_0]$ of the typical cylinder. In addition, we assume that $R_0 \sim F$ and $\Phi_0 \sim G$ are independent, that is, $\mathbf{P}(R_0 \leqslant r, \Phi_0 \leqslant \varphi) = F(r)G(\varphi)$. Now we introduce a stationary independently marked PP as a locally finite simple counting measure $\Psi_{F,G}^P := \sum_{i\in\mathbb{Z}} \delta_{[P_i,(\Phi_i,R_i)]}$ defined on the Borel sets of $\mathbb{R}^1 \times [0,\pi) \times [0,\infty)$ whose finite-dimensional distributions are shift-invariant in the first component; see, for example, [2, 6] or [19]. The stationary ground PP $\Psi = \sum_{i\in\mathbb{Z}} \delta_{P_i} \sim P$ with finite and positive intensity $\lambda := \mathbf{E}\,\Psi([0,1]) > 0$ is assumed to be independent of the i.i.d. sequence $\{(\Phi_i, R_i)\colon i \in \mathbb{Z} := \{0, \pm1, \pm2, \dots\}\}$ of mark vectors. Then the intensity measure $\Lambda_{F,G}((\cdot)\times[0,\varphi]\times[0,r]) := \mathbf{E}\,\Psi_{F,G}^P((\cdot)\times[0,\varphi]\times[0,r])$ of $\Psi_{F,G}^P$ can be expressed for $r \geqslant 0$ and $0 \leqslant \varphi \leqslant \pi$ as

$$\Lambda_{F,G}\big((\cdot)\times[0,\varphi]\times[0,r]\big) = \mathbf{E}\Psi(\cdot)\,\mathbf{P}(\Phi_0 \leqslant \varphi,\ R_0 \leqslant r) = \lambda|\cdot|_1 G(\varphi)F(r) \quad \text{with } \lambda > 0,$$

where $|\cdot|_k$ denotes the Lebesgue measure on $\mathbb{R}^k$. Each triplet $[P_i,(\Phi_i,R_i)]$, $i \in \mathbb{Z}$, determines a *random cylinder* $g(P_i,\Phi_i) \oplus b(\mathbf{o},R_i)$, where $b(\mathbf{o},r)$ is the circle in $\mathbb{R}^2$ with radius $r \geqslant 0$ and center in the origin $\mathbf{o}$, $\oplus$ stands for pointwise addition (Minkowski sum) of sets in $\mathbb{R}^2$, and $g(p,\varphi) := \{(x,y) \in \mathbb{R}^2\colon x\cos\varphi + y\sin\varphi = p\}$ denotes the unique line with signed distance $p \in \mathbb{R}^1$ from $\mathbf{o}$ and an angle $\varphi \in [0,\pi)$ measured anticlockwise between the x-axis and normal vector $v(\varphi) = (\cos\varphi, \sin\varphi)$ on the line with direction in the half-plane not containing $\mathbf{o}$. Now we are in a position to define the main subject of this paper.

DEFINITION 1. A CP $\Xi = \Xi_{F,G}^P$ in the Euclidean plane $\mathbb{R}^2$ derived from the stationary independently marked PP $\Psi_{F,G}^P$ is defined by the random union set

$$\Xi_{F,G}^P := \bigcup_{i\in\mathbb{Z}} \big(g(P_i,\Phi_i) \oplus b(\mathbf{o},R_i)\big). \tag{1.1}$$

Our first aim is to prove the mean-square convergence (and thus the convergence in probability) of the ratio $|\Xi \cap \varrho K|_2/|\varrho K|_2$ to the deterministic limit $1 - \exp\{-\lambda\mathbf{E}|\Xi_0|_1\}$ as $\varrho \to \infty$ for a compact star-shaped set $K \subset \mathbb{R}^2$ with respect to the origin $\mathbf{o}$, an inner point of K; see Theorem 1. The Brillinger-mixing condition put on the ground PP $\Psi \sim P$ is essential to obtain this result. Our second main result, Theorem 2, shows the existence and explicit shape of the asymptotic variance

$$\lim_{\varrho\to\infty} \varrho^{-3}\,\mathrm{Var}\,|\Xi \cap \varrho K|_2 =: \sigma_P^2(K,F,G). \tag{1.2}$$

Theorems 1 and 2 generalize some of the results obtained in [13] and [14] (in particular, Theorem 2 in [14]) for Poisson CPs in $\mathbb{R}^d$ to planar CPs generated by Brillinger-mixing PPes.

The limit $\sigma_P^2(K,F,G)$ is positive and finite (if $\mathbf{E}|\Xi_0|_1^2 = 4\mathbf{E}R_0^2 < \infty$) and depends on the shape of K, the intensity λ, the first and second moments of F, and the distribution function G, which is assumed to be continuous (not necessarily absolutely continuous). A purely discrete distribution function G yields different expressions for $\sigma_P^2(K,F,G)$ even if $\Psi \sim P = \Pi_\lambda$ is a stationary Poisson PP with intensity $\lambda > 0$; see [13, 14]. A distribution function G without jumps implies that $\mathbf{P}(\Phi_0 = \Phi_1) = 0$ if the angles $\Phi_0, \Phi_1 \sim G$ are independent. Note that the order ϱ^3 of the growth of $\mathrm{Var}\,|\Xi \cap \varrho K|_2$ is much faster than the growth of the area $|\varrho K|_2 = \varrho^2|K|_2$, which reveals a typical feature of long-range dependence within the random set (1.1) and in general need not be closed or stationary.

2 Basic assumptions and main results

Recall that the kth-order *factorial cumulant measure* $\gamma^{(k)}(\cdot)$ of a PP $\Psi \sim P$ for $k \in \mathbb{N}$ is defined by

$$\gamma^{(k)}\Big(\mathop{\times}_{i=1}^{k} B_i\Big) := \sum_{\ell=1}^{k}(-1)^{\ell-1}(\ell-1)! \sum_{K_1\cup\cdots\cup K_\ell=\{1,\ldots,k\}} \prod_{j=1}^{\ell} \alpha^{(\#K_j)}\Big(\mathop{\times}_{i\in K_j} B_i\Big), \tag{2.1}$$

where $\alpha^{(k)}$ denotes the kth-order *factorial moment measure* of $\Psi \sim P$ defined by

$$\alpha^{(k)}\Big(\mathop{\times}_{j=1}^{k} B_j\Big) := \mathbf{E}\Big(\sum_{i_1,\ldots,i_k\in\mathbb{Z}}^{\neq} \mathbf{1}_{B_1}(P_{i_1})\cdots \mathbf{1}_{B_k}(P_{i_k})\Big)$$

for bounded Borel sets $B_1,\ldots,B_k \subset \mathbb{R}^1$, where the sum $\sum^{\neq}$ runs over k-tuples of pairwise distinct integers. Formula (2.1) is based on the general relationship between mixed moments and mixed cumulants; see [2] or [18]. Note that $\gamma^{(k)}$ is a locally finite signed measure on $[\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)]$.

Due to the stationarity of $\Psi \sim P$, we may define the kth-order *reduced cumulant measure* $\gamma_{\mathrm{red}}^{(k)}$ as the unique signed measure on $[\mathbb{R}^{k-1}, \mathcal{B}(\mathbb{R}^{k-1})]$ satisfying

$$\gamma^{(k)}\Big(\mathop{\times}_{i=1}^{k} B_i\Big) = \lambda \int_{B_1} \gamma_{\mathrm{red}}^{(k)}\Big(\mathop{\times}_{i=2}^{k}(B_i - p)\Big)\,\mathrm{d}p$$

for all bounded sets $B_1,\ldots,B_k \in \mathcal{B}(\mathbb{R}^1)$. The *total-variation measure* $|\gamma_{\mathrm{red}}^{(k)}|$ is defined by $|\gamma_{\mathrm{red}}^{(k)}| = (\gamma_{\mathrm{red}}^{(k)})^+ + (\gamma_{\mathrm{red}}^{(k)})^-$, where the measures $(\gamma_{\mathrm{red}}^{(k)})^+$ and $(\gamma_{\mathrm{red}}^{(k)})^-$ are given by the Jordan decomposition of the signed measure $\gamma_{\mathrm{red}}^{(k)} = (\gamma_{\mathrm{red}}^{(k)})^+ - (\gamma_{\mathrm{red}}^{(k)})^-$. The *total variation* of $\gamma_{\mathrm{red}}^{(k)}$ on $[\mathbb{R}^{k-1}, \mathcal{B}(\mathbb{R}^{k-1})]$ is defined as $\|\gamma_{\mathrm{red}}^{(k)}\|_{TV} := |\gamma_{\mathrm{red}}^{(k)}|(\mathbb{R}^{k-1})$.

Furthermore, if $\gamma_{\mathrm{red}}^{(k)}$ possesses a Lebesgue density $c_{\mathrm{red}}^{(k)}$ on $\mathbb{R}^{k-1}$ (called the kth-order *reduced cumulant density*), then we need the usual L_q-norm $\|c_{\mathrm{red}}^{(k)}\|_q := (\int_{\mathbb{R}^{k-1}} |c_{\mathrm{red}}^{(k)}(x)|^q\,\mathrm{d}x)^{1/q}$ for $k \geqslant 2$, the modified L_q^*-norm $\|c_{\mathrm{red}}^{(k)}\|_q^* := \int_{\mathbb{R}^1}(\int_{\mathbb{R}^{k-2}} |c_{\mathrm{red}}^{(k)}(x,p)|^q\,\mathrm{d}x)^{1/q}\,\mathrm{d}p$ for $k \geqslant 3$, and $\|c_{\mathrm{red}}^{(2)}\|_q^* := \|c_{\mathrm{red}}^{(2)}\|_q$, where $1 \leqslant q < \infty$. Formally, we may put $\|\gamma_{\mathrm{red}}^{(1)}\|_{TV} := 1$ and $\|c_{\mathrm{red}}^{(1)}\|_q = \|c_{\mathrm{red}}^{(1)}\|_q^* := 1$. Note that the existence and integrability of $c_{\mathrm{red}}^{(k)}$ imply that $\|c_{\mathrm{red}}^{(2)}\|_1 = \|\gamma_{\mathrm{red}}^{(2)}\|_{TV}$ and $\|c_{\mathrm{red}}^{(k)}\|_1 = \|c_{\mathrm{red}}^{(k)}\|_1^* = \|\gamma_{\mathrm{red}}^{(k)}\|_{TV}$ for all $k \geqslant 3$.

DEFINITION 2. A stationary PP $\Psi \sim P$ on $[\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1)]$ with intensity $\lambda > 0$ satisfying $\mathbf{E}\Psi^k([0,1]) < \infty$ for all $k \geqslant 2$ is called

(i) *Brillinger mixing* if $\|\gamma_{\mathrm{red}}^{(k)}\|_{TV} < \infty$ for all $k \geqslant 2$,

(ii) *strongly Brillinger mixing* if there are constants $b > 0$ and $a \geqslant b^{-1}$ such that $\|\gamma_{\mathrm{red}}^{(k)}\|_{TV} \leqslant ab^k k!$,

(iii) *strongly L_q-Brillinger mixing* (resp., *strongly L_q^*-Brillinger mixing*) for some $q \geqslant 1$ if there exists $c_{\mathrm{red}}^{(k)}$ such that $\|c_{\mathrm{red}}^{(2)}\|_1 < \infty$ and $\|c_{\mathrm{red}}^{(k)}\|_q \leqslant a_q\,(b_q)^k\,k!$ for $k \geqslant 2$ with constants $b_q > 0$ and $a_q \geqslant (b_q)^{-1}$ (resp., $\|c_{\mathrm{red}}^{(k)}\|_q^* \leqslant a_q^*\,(b_q^*)^k\,k!$ for $k \geqslant 2$ with constants $b_q^* > 0$ and $a_q^* \geqslant (b_q^*)^{-1}$).

Remark 1. In general, the Brillinger-mixing condition is formulated for stationary PPs on $\mathbb{R}^d, d \geqslant 1$. This condition expresses some kind of mutual asymptotic uncorrelatedness of the numbers of points in bounded sets with unboundedly increasing distance from each other. This type of weak dependence does not necessarily imply ergodicity (see [9]), but allows us to prove central limit theorems for various stochastic models related to point processes, for example, in stochastic geometry, statistical physics for $d \geqslant 1$, or in queueing theory for $d = 1$; see, for example, [12]. In [10, 11] the relations between (strong) Brillinger-mixing and classical mixing conditions are studied. Strong Brillinger mixing requires exponential moments of the number of points in bounded sets. For any dimension $d \geqslant 1$, examples of such point processes are determinantal point processes (see [8]), Poisson cluster processes if the number of daughter points has an exponential moment, certain Cox processes, and Gibbsian PPs under suitable restrictions; see, for example, [17]. For $d = 1$, renewal processes with an exponentially decaying interrenewal density (see [12]) and, among them, the Erlang and Macchi processes (see [2, p. 144]) are strongly Brillinger mixing.

Our first result can be regarded as a mean-square ergodic theorem for the random set (1.1).

Theorem 1. *Assume that a stationary PP $\Psi \sim P$ on $\mathbb{R}^1$ is Brillinger mixing. Further suppose that $\mathbf{E}R_0 < \infty$ and $\Phi_0 \sim G$ has a continuous distribution function G. Then*

$$\frac{|\Xi \cap \varrho K|_2}{|\varrho K|_2} \underset{\varrho\to\infty}{\overset{L^2(\mathbf{P})}{\longrightarrow}} 1 - \exp\{-\lambda \mathbf{E}|\Xi_0|_1\}, \tag{2.2}$$

which immediately implies the convergence in probability of the ratio $|\Xi \cap \varrho K|_2/|\varrho K|_2$ and its $L^p(\mathbf{P})$-convergence for any $p \geqslant 1$. The limit (2.2) *is the same as that for the Poisson PP $\Psi \sim \Pi_\lambda$.*

Our second result provides an exact asymptotic behavior of the variance of the area of the cylinder process (1.1) that is contained in a star-shaped set ϱK growing unboundedly in all directions. For this purpose, in comparison with Theorem 1, we need a strengthening and quantification of the usual Brillinger-mixing condition.

Theorem 2. *Assume that the stationary PP $\Psi \sim P$ on $\mathbb{R}^1$ is either strongly Brillinger mixing with $b < 1/2$ or strongly L_q-Brillinger mixing with $(\mathbf{E}|\Xi_0|_1)^{1-1/q} b_q < 1/2$ or strongly L_q^*-Brillinger mixing with $(\mathbf{E}|\Xi_0|_1)^{1-1/q} b_q^* < 1/2$ for some $q > 1$. Further suppose that $\mathbf{E}R_0^2 < \infty$ and $\Phi_0 \sim G$ has a continuous distribution function G. Then the limit* (1.2) *is positive and finite with*

$$\sigma_P^2(K, F, G) = \lambda \mathrm{e}^{-2\lambda \mathbf{E}|\Xi_0|_1} \big((\mathbf{E}|\Xi_0|_1)^2 \gamma_{\mathrm{red}}^{(2)}(\mathbb{R}^1) C_1^{G,K} + 2\mathbf{E}|\Xi_0|_1^2 C_2^{G,K} \big), \tag{2.3}$$

where

$$C_1^{G,K} := \int\limits_{\mathbb{R}^1} \big(\mathbf{E}\big|g(p,\Phi_0) \cap K\big|_1\big)^2 \,\mathrm{d}p \quad \textit{and} \quad C_2^{G,K} := \int\limits_0^{\pi} \int\limits_0^{r_K(\varphi\pm\pi/2)} \left|K \cap \left(K + s v\left(\varphi \pm \frac{\pi}{2}\right)\right)\right|_2 \mathrm{d}s \,\mathrm{d}G(\varphi)$$

with $r_K(\psi) := \max\{r \geqslant 0\colon r v(\psi) \in K \oplus (-K)\}$, where $r_K(\psi) = r_K(\psi \pm \pi)$ for reasons of symmetry.

Remark 2. In the particular case $K = b(\mathbf{o}, 1)$, we can show that $C_1^{G,K} = 16/3$ and $C_2^{G,K} = 8/3$ are independent of the distribution function G. If Φ_0 is uniformly distributed on $[0, \pi]$, then we get

$$C_1^{G,K} = \frac{1}{\pi^2} \int\limits_{\mathbb{R}^1} \left(\int\limits_0^{\pi} \big|g(p,\varphi) \cap K\big|_1 \,\mathrm{d}\varphi \right)^2 \mathrm{d}p \quad \text{and} \quad C_2^{G,K} = \frac{1}{2\pi} \int\limits_K \int\limits_K \frac{\mathrm{d}x \,\mathrm{d}y}{\|x - y\|}.$$

The latter double integral is known as the *second-order chord power integral of K*; see, for example, [14, p. 327] and [19, Chap. 7] for integral geometric background.

The proofs of our results are based on series expansions of expectation and variance of the area $|\Xi \cap \varrho K|_2$ in terms of the factorial moment and cumulant measures of the PP $\Psi \sim P$; see [2, Chap. 5.5]. These expansions and their convergence are studied in Section 3 using the above-defined Brillinger-mixing conditions. The asymptotics of these expansions (as $\varrho \to \infty$) are studied in Sections 4 and 5. Note that the study of non-Poisson CPs is much more complex and difficult than that of Poisson CPs. To emphasize the basic ideas on the one hand and to keep our mathematical tools as simple as possible on the other hand, we study only planar CPs. In the proofs of our results, we have shortened some longer calculations at certain points. For more detailed derivations, we refer the interested readers to [5].

3 Factorial moment expansions of $\mathrm{E}|\Xi \cap \varrho K|_2$ and $\mathrm{Var}\,|\Xi \cap \varrho K|_2$

The distribution of a random closed set Ξ is determined by its *Choquet functional* (see, e.g., [15] or [16])

$$T_\Xi(X) := \mathbf{P}(\Xi \cap X \neq \emptyset) \quad \text{for } X \in \mathcal{K}_2,$$

where $\mathcal{K}_2$ denotes the family of nonempty compact sets in $\mathbb{R}^2$. The following lemma shows the connection between the Choquet functional and the *probability generating functional* (PGF) $G_P[w(\cdot)]$ of $\Psi \sim P$ defined for Borel-measurable functions $w : \mathbb{R}^1 \to [0,1]$ satisfying $\int_{\mathbb{R}^1}(1-w(x))\,\mathrm{d}x < \infty$ by

$$G_P\big[w(\cdot)\big] := \mathbf{E}\Bigg(\prod_{i:\,\Psi(\{P_i\})>0} w(P_i)\Bigg) = \int_{\mathbf{N}} \prod_{p\in\mathbb{R}^1:\,\psi(\{p\})>0} w(p)\,P(\mathrm{d}\psi), \tag{3.1}$$

where $\mathbf{N}$ denotes the set of locally finite simple counting measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^1)$.

Lemma 1. *For any $X \in \mathcal{K}_2$, we have the representation*

$$T_\Xi(X) = 1 - G_P\big[1 - \mathbf{P}\big((\cdot) \in [-R_0, R_0] \oplus \langle v(\Phi_0), X\rangle\big)\big], \tag{3.2}$$

where $\langle v(\Phi_0), X\rangle := \bigcup_{x\in X}\langle v(\Phi_0), x\rangle$ with $\langle v(\varphi), x\rangle = x^{(1)}\cos\varphi + x^{(2)}\sin\varphi$ for $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^2$.

To simplify the notation, for $k \geqslant 2$ (not necessarily distinct) points $x_1, \ldots, x_k \in \mathbb{R}^2$, we define

$$w^{\cup}_{x_1,\ldots,x_k}(p) := \mathbf{P}\Bigg(p \in \bigcup_{i=1}^{k}\big(\Xi_0 + \langle v(\Phi_0), x_i\rangle\big)\Bigg) \quad \text{and} \quad w^{\cap}_{x_1,\ldots,x_k}(p) := \mathbf{P}\Bigg(p \in \bigcap_{i=1}^{k}\big(\Xi_0 + \langle v(\Phi_0), x_i\rangle\big)\Bigg).$$

For brevity, put $w_x(p) := w^{\cup}_x(p) = w^{\cap}_x(p)$. Obviously, $w^{\cup}_{x_1,x_2}(p) = w_{x_1}(p) + w_{x_2}(p) - w^{\cap}_{x_1,x_2}(p)$.

Corollary 1. *For $X = \{x_1, \ldots, x_k\}$ with pairwise distinct points $x_1 \ldots, x_k \in \mathbb{R}^2$, we have*

$$\mathbf{P}\big(x_1 \in \Xi^c, \ldots, x_k \in \Xi^c\big) = 1 - T_\Xi\big(\{x_1, \ldots, x_k\}\big) = G_P\big[1 - w^{\cup}_{x_1,\ldots,x_k}(\cdot)\big]. \tag{3.3}$$

Proof of Lemma 1. To prove formula (3.2), we use the orthogonal matrix

$$O(\varphi) = \begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}, \tag{3.4}$$

which represents an anticlockwise rotation by the angle $\varphi \in [0,\pi)$, so that we have $v(\varphi)O(\varphi) = (1,0)$ and $v(\varphi) = (1,0)O(-\varphi)$, since $O(-\varphi) = O^T(\varphi) = O^{-1}(\varphi)$. Using the PGF (3.1) and the independence assumption in the definition of (1.1), we obtain

$$1 - T_{\Xi}(X) = \mathbf{P}(\Xi \cap X = \emptyset) = \mathbf{P}\Bigg(\bigcap_{i:\, \Psi(\{P_i\})>0} \big\{ \big(g(P_i, \Phi_i) \oplus b(\mathbf{o}, R_i)\big) \cap X = \emptyset \big\} \Bigg)$$
$$= \int_{\mathbf{N}} \prod_{p \in \mathbb{R}^1:\, \psi(\{p\})>0} \mathbf{P}\big(\big(g(p, \Phi_0) \oplus b(\mathbf{o}, R_0)\big) \cap X = \emptyset \bigm| \Psi = \psi \big)\, P(\mathrm{d}\psi)$$
$$= \int_{\mathbf{N}} \prod_{p \in \mathbb{R}^1:\, \psi(\{p\})>0} \big(1 - \mathbf{P}\big(p \in [-R_0, R_0] \oplus \big\langle v(\Phi_0), X \big\rangle\big)\big)\, P(\mathrm{d}\psi). \tag{3.5}$$

To verify (3.5), we use the fact that $x \in g(p, \varphi) \oplus b(\mathbf{o}, r)$ iff $p \in [-r, r] + \langle v(\varphi), x \rangle$, which implies

$$\big\{ \big(g(p, \Phi_0) \oplus b(\mathbf{o}, R_0)\big) \cap X \neq \emptyset \big\} = \big\{ p \in [-R_0, R_0] \oplus \big\langle v(\Phi_0), X \big\rangle \big\}.$$

Obviously, (3.5) coincides with (3.2). Hence the proof of Lemma 1 is complete. □

We recall the fact that the probability space $[\Omega, \mathcal{F}, \mathbf{P}]$ on which the marked point process $\Psi^P_{F,G}$ is defined can be chosen in such a way that the mapping $(x, \omega) \mapsto \mathbf{1}_{\Xi(\omega)}(x) \in \{0, 1\}$ for $(x, \omega) \in \mathbb{R}^2 \times \Omega$ is measurable with respect to the product σ-algebra $\mathcal{B}(\mathbb{R}^2) \otimes \mathcal{F}$; see Appendix in [7]. This enables us to apply Fubini's theorem to the family of indicator variables $\{\mathbf{1}_{\Xi}(x),\ x \in \mathbb{R}^2\}$, so that the kth-order *mixed moment function*

$$p^{(k)}_{\Xi}(x_1, \dots, x_k) := \mathbf{E}\Bigg(\prod_{j=1}^{k} \mathbf{1}_{\Xi}(x_j) \Bigg) = \mathbf{P}(x_1 \in \Xi,\ \dots,\ x_k \in \Xi), \quad x_1, \dots, x_k \in \mathbb{R}^2, \tag{3.6}$$

is $\mathcal{B}(\mathbb{R}^{2k})$-measurable for $k \in \mathbb{N} := \{1, 2, \dots\}$. The mixed moment functions $p^{(k)}_{\Xi^c}(x_1, \dots, x_k)$ of the random field $\{\mathbf{1}_{\Xi^c}(x),\ x \in \mathbb{R}^2\}$ are represented in (3.3) in terms of T_{Ξ} and the PGF (3.1).

Let us fix a star-shaped set $K \in \mathcal{K}_2$ containing the origin $\mathbf{o}$ as an inner point. Further, let $\varrho \geqslant 1$ be a scaling factor tending to infinity implying that $\varrho K \uparrow \mathbb{R}^2$ as $\varrho \to \infty$. The second-order mixed moment functions (3.6) fulfill the relation

$$p^{(2)}_{\Xi}(x_1, x_2) - p^{(1)}_{\Xi}(x_1) p^{(1)}_{\Xi}(x_2) = p^{(2)}_{\Xi^c}(x_1, x_2) - p^{(1)}_{\Xi^c}(x_1) p^{(1)}_{\Xi^c}(x_2).$$

By applying Fubini's theorem together with (3.2) and (3.4) we get that

$$\mathbf{E}|\Xi \cap \varrho K|_2 = \int_{\varrho K} p^{(1)}_{\Xi}(x)\, \mathrm{d}x = \varrho^2 \int_K T_{\Xi}\big(\{\varrho x\}\big)\, \mathrm{d}x = \varrho^2 \int_K \big(1 - G_P\big[1 - w_{\varrho x}(\cdot)\big]\big)\, \mathrm{d}x. \tag{3.7}$$

By Corollary 1, together with the equality

$$\operatorname{Var}|\Xi \cap \varrho K|_2 = \int_{\varrho K} \int_{\varrho K} \big(p^{(2)}_{\Xi^c}(x_1, x_2) - p^{(1)}_{\Xi^c}(x_1) p^{(1)}_{\Xi^c}(x_2) \big)\, \mathrm{d}x_1\, \mathrm{d}x_2,$$

we obtain the following:

Lemma 2. *We have*

$$\operatorname{Var}|\Xi \cap \varrho K|_2 = \varrho^4 \int_K \int_K \Bigg(G_P\big[1 - w^{\cup}_{\varrho x_1, \varrho x_2}(\cdot)\big] - \prod_{i=1}^{2} G_P\big[1 - w_{\varrho x_i}(\cdot)\big] \Bigg)\, \mathrm{d}x_1\, \mathrm{d}x_2. \tag{3.8}$$

As a consequence of (3.7) and the definition of the factorial moment measures $\alpha^{(k)}$ of Ψ (see [3, Chap. 9.5]), we get the following series expansion:

$$\mathbf{E}|\Xi \cap \varrho K|_2 = \varrho^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int\limits_K \int\limits_{\mathbb{R}^k} \prod_{j=1}^{k} w_{\varrho x}(p_j)\, \alpha^{(k)}(\mathrm{d}\mathbf{p}_{1,k})\, \mathrm{d}x, \tag{3.9}$$

provided that the series on the right-hand side is convergent, where $\mathbf{p}_{i,k} := (p_i, \dots, p_k)$ for $k \geqslant i$ and $i = 1$ or $i = 2$. By the Bonferroni inequalities (see [3, Prop. 9.5.VI]) it follows that

$$\left| 1 - G_P\big[1 - w_{\varrho x}(\cdot)\big] - \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k!} \int\limits_{\mathbb{R}^k} \prod_{j=1}^{k} w_{\varrho x}(p_j)\, \alpha^{(k)}(\mathrm{d}\mathbf{p}_{1,k}) \right| \leqslant \frac{1}{m!} \int\limits_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\varrho x}(p_j)\, \alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m}) \tag{3.10}$$

for all $m \geqslant 1$ and $x \in \mathbb{R}^2$. Consequently, the right-hand side of (3.9) is convergent if and only if

$$\frac{1}{m!} \int\limits_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\varrho x}(p_j)\, \alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m}) \underset{m\to\infty}{\longrightarrow} 0.$$

To show this convergence, we express $\alpha^{(m)}$ by the *factorial cumulant measures* $\gamma^{(k)}, k = 1, \dots, m$, where $\gamma^{(1)}(B) = \alpha^{(1)}(B) = \lambda |B|_1$ and

$$\alpha^{(k)}\Big(\mathop{\times}_{i=1}^{k} B_i\Big) = \sum_{\ell=1}^{k} \sum_{K_1 \cup \dots \cup K_\ell = \{1,\dots,k\}} \prod_{j=1}^{\ell} \gamma^{(\# K_j)}\Big(\mathop{\times}_{i \in K_j} B_i\Big) \quad \text{for } k \geqslant 2. \tag{3.11}$$

Representation (3.11) follows by inverting formula (2.1). This gives us a tool to prove the following:

Lemma 3. *If a stationary PP $\Psi \sim P$ is strongly Brillinger mixing with $b < 1/2$ and $\mathbf{E}R_0 < \infty$, then*

$$\frac{1}{m!} \int\limits_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\varrho x}(p_j)\, \alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m}) \leqslant \frac{1}{2}(2b)^m \big(\exp\big\{a\lambda \mathbf{E}|\Xi_0|_1\big\} - 1\big) \underset{m\to\infty}{\longrightarrow} 0, \tag{3.12}$$

which, together with (3.10)*, implies* (3.9)*. If $\Psi \sim P$ is strongly L_q-(L_q^*)-Brillinger mixing for some $q > 1$ such that $(\mathbf{E}|\Xi_0|_1)^{1-1/q} b_q < 1/2$ ($(\mathbf{E}|\Xi_0|_1)^{1-1/q} b_q^* < 1/2$), then estimate* (3.12) *remains valid with a and b replaced by $a_q(\mathbf{E}|\Xi_0|_1)^{1/q-1}$ ($a_q^*(\mathbf{E}|\Xi_0|_1)^{1/q-1}$) and $b_q(\mathbf{E}|\Xi_0|_1)^{1-1/q}$ ($b_q^*(\mathbf{E}|\Xi_0|_1)^{1-1/q}$), respectively.*

Proof of Lemma 3. Using representation (3.11), we obtain

$$\begin{aligned}
&\frac{1}{m!} \int\limits_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\varrho x}(p_j)\, \alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m}) \\
&= \frac{1}{m!} \sum_{\ell=1}^{m} \sum_{K_1 \cup \dots \cup K_\ell = \{1,\dots,m\}} \prod_{j=1}^{\ell} \int\limits_{\mathbb{R}^{\# K_j}} \prod_{i \in K_j} w_{\varrho x}(p_i)\, \gamma^{(\# K_j)}\big(\mathrm{d}(p_i \colon i \in K_j)\big) \\
&= \frac{1}{m!} \sum_{\ell=1}^{m} \frac{1}{\ell!} \sum_{\substack{k_1 + \dots + k_\ell = m \\ k_i \geqslant 1,\ i=1\dots,\ell}} \frac{m!}{k_1! \cdots k_\ell!} \prod_{j=1}^{\ell} f(k_j) = \sum_{\ell=1}^{m} \frac{1}{\ell!} \sum_{\substack{k_1 + \dots + k_\ell = m \\ k_i \geqslant 1,\ i=1\dots,\ell}} \prod_{i=1}^{\ell} \frac{f(k_i)}{k_i!},
\end{aligned} \tag{3.13}$$

where

$$f(k) := \int\limits_{\mathbb{R}^k} \prod_{i=1}^{k} w_{\varrho x}(p_i)\, \gamma^{(k)}(\mathrm{d}\mathbf{p}_{1,k}) = \lambda \int\limits_{\mathbb{R}^1} w_{\varrho x}(p_1) \int\limits_{\mathbb{R}^{k-1}} \prod_{i=2}^{k} w_{\varrho x}(p_i + p_1)\, \gamma_{\mathrm{red}}^{(k)}(\mathrm{d}\mathbf{p}_{2,k})\, \mathrm{d}p_1$$

for $k = 1, \dots, m$ and fixed $\varrho \geqslant 1$ and $x \in \mathbb{R}^2$. Equality (3.13) is justified by the invariance of $\gamma^{(k)}(\times_{i=1}^{k} B_i)$ under permutations of the sets $B_1, \dots, B_k \in \mathcal{B}(\mathbb{R}^1)$ for any $k \in \mathbb{N}$. We proceed with

$$\bigl|f(k)\bigr| \leqslant \lambda \int\limits_{\mathbb{R}^1} w_{\varrho x}(p_1) \int\limits_{\mathbb{R}^{k-1}} \bigl|\gamma_{\mathrm{red}}^{(k)}\bigr|(\mathrm{d}\mathbf{p}_{2,k})\, \mathrm{d}p_1 = \lambda \mathbf{E}|\Xi_0|_1 \bigl\|\gamma_{\mathrm{red}}^{(k)}\bigr\|_{TV} \leqslant \lambda \mathbf{E}|\Xi_0|_1 a b^k k!$$

for all $k \in \mathbb{N}$. Here we have used Fubini's theorem combined with $w_{\varrho x}(p) \leqslant 1$ for $p \in \mathbb{R}^1$, so that

$$\int\limits_{\mathbb{R}^1} w_{\varrho x}(p)\, \mathrm{d}p = \int\limits_{\mathbb{R}^1} \mathbf{P}\bigl(p \in \Xi_0 + \varrho\langle v(\Phi_0),\, x\rangle\bigr)\, \mathrm{d}p = \int\limits_{\mathbb{R}^1} \mathbf{P}\bigl(p \in \Xi_0\bigr)\, \mathrm{d}p = \mathbf{E}|\Xi_0|_1.$$

Hence, together with some elementary combinatorics, we arrive at

$$\sum_{\ell=1}^{m} \frac{1}{\ell!} \sum_{\substack{k_1+\dots+k_\ell=m \\ k_i \geqslant 1,\, i=1\dots,\ell}} \prod_{i=1}^{\ell} \frac{|f(k_i)|}{k_i!} \leqslant b^m \sum_{\ell=1}^{m} \frac{(a\lambda \mathbf{E}|\Xi_0|_1)^\ell}{\ell!} \binom{m-1}{\ell-1} \leqslant b^m 2^{m-1} \bigl(\exp\bigl\{a\lambda \mathbf{E}|\Xi_0|_1\bigr\} - 1\bigr).$$

Combining (3.13) with the latter bound for $b < 1/2$ immediately leads to estimate (3.12). Under the strong L_q-Brillinger-mixing condition, we may express $f(k)$ for $k \geqslant 2$ as follows:

$$f(k) = \lambda \int\limits_{\mathbb{R}^1} w_{\varrho x}(p_1) \mathbf{E} \int\limits_{\mathbb{R}^{k-1}} \prod_{i=2}^{k} \mathbf{1}_{\Xi_i + \varrho\langle v(\Phi_i),\, x\rangle}(p_i + p_1) c_{\mathrm{red}}^{(k)}(\mathbf{p}_{2,k})\, \mathrm{d}\mathbf{p}_{2,k}\, \mathrm{d}p_1,$$

where $\Xi_i = [-R_i, R_i]$, and $(R_2, \Phi_2), \dots, (R_k, \Phi_k)$ are i.i.d. random vectors with the same distribution as $(R_0.\Phi_0)$. Applying Hölder's inequality for $q > 1$ and $p = q/(q-1)$, Lyapunov's inequality $\mathbf{E}|\Xi_0|^{1/p} \leqslant (\mathbf{E}|\Xi_0|)^{1/p} = (\mathbf{E}|\Xi_0|)^{1-1/q}$, and the condition $\|c_{\mathrm{red}}^{(k)}\|_q \leqslant a_q (b_q)^k k!$, we obtain that

$$\begin{aligned} \bigl|f(k)\bigr| &\leqslant \lambda \bigl\|c_{\mathrm{red}}^{(k)}\bigr\|_q \mathbf{E}|\Xi_1|_1 \prod_{i=2}^{k} \mathbf{E}|\Xi_i|_1^{1/p} \leqslant \lambda \bigl\|c_{\mathrm{red}}^{(k)}\bigr\|_q \bigl(\mathbf{E}|\Xi_0|_1\bigr)^{1+(k-1)/p} \\ &\leqslant \lambda a_q \bigl(\mathbf{E}|\Xi_0|_1\bigr)^{1/q} \bigl(b_p \bigl(\mathbf{E}|\Xi_0|_1\bigr)^{1-1/q}\bigr)^k k! \end{aligned}$$

for all $k \in \mathbb{N}$. The latter estimate with a_q^* and b_q^* instead of a_q and b_q can be shown under the strong L_q^*-Brillinger-mixing condition. The details are omitted. Finally, we have to repeat the foregoing steps with the latter bound for $f(k)$, which completes the proof of Lemma 3. □

4 Some auxiliary lemmas

The following Lemmas 4–6 are essential for the calculation of the terms on the right-hand side of (2.3). It is interesting that the assumptions in these lemmas are rather mild in comparison with the Brillinger-mixing-type conditions in Theorems 1 and 2.

Lemma 4. *Let $\Psi \sim P$ be a stationary PP on $\mathbb{R}^1$ satisfying $\|\gamma^{(k)}_{\mathrm{red}}\|_{TV} < \infty$ for $k = 2, \dots, m$, where $m \geqslant 2$ is fixed. If $\mathbf{E}R_0 < \infty$ and $\Phi_0 \sim G$ has a continuous distribution function G, then for (not necessarily distinct) points $x_1, \dots, x_m \in \mathbb{R}^2 \setminus \{\mathbf{o}\}$,*

$$\int_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\varrho x_j}(p_j)\, \alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m}) \underset{\varrho\to\infty}{\longrightarrow} \big(\lambda \mathbf{E}|\Xi_0|_1\big)^m. \tag{4.1}$$

From Lemma 4 and (3.10) we obtain the limit of the ratio $\mathbf{E}|\Xi \cap \varrho K|_2 / |\varrho K|_2$ as $\varrho \to \infty$.

Corollary 2. *Let $\Psi \sim P$ be a Brillinger-mixing PP on $\mathbb{R}^1$. If $\mathbf{E}R_0 < \infty$ and $\Phi_0 \sim G$ has a continuous distribution function G, then*

$$\frac{\mathbf{E}|\Xi \cap \varrho K|_2}{|\varrho K|_2} \underset{\varrho\to\infty}{\longrightarrow} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \big(\lambda \mathbf{E}|\Xi_0|_1\big)^k = 1 - \exp\big\{-\lambda \mathbf{E}|\Xi_0|_1\big\}.$$

Proof of Corollary 2. Relation (4.1) for $x_1 = \dots = x_m = x \neq \mathbf{o}$ applied to inequality (3.9) yields

$$\left| \liminf_{\varrho\to\infty} \big(1 - G_P\big[1 - w_{\varrho x}(\cdot)\big]\big) - \sum_{k=1}^{m-1} \frac{(-1)^{k-1}\big(\lambda \mathbf{E}|\Xi_0|_1\big)^k}{k!} \right| \leqslant \frac{(\lambda \mathbf{E}|\Xi_0|_1)^m}{m!} \quad \text{for any } m \geqslant 1.$$

This inequality is also valid if $\liminf_{\varrho\to\infty}$ is replaced by $\limsup_{\varrho\to\infty}$. Letting $m \to \infty$ shows that $\lim_{\varrho\to\infty}(1 - G_P[1 - w_{\varrho x}(\cdot)])$ exists, and combining this with (3.7) reveals that

$$\frac{\mathbf{E}|\Xi \cap \varrho K|_2}{|\varrho K|_2} = \frac{1}{|K|_2} \int_K \big(1 - G_P\big[1 - w_{\varrho x}(\cdot)\big]\big)\, \mathrm{d}x \underset{\varrho\to\infty}{\longrightarrow} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\big(\lambda \mathbf{E}|\Xi_0|_1\big)^k}{k!},$$

which immediately yields Corollary 2. □

Lemma 5. *Let Ψ be a second-order stationary PP on $\mathbb{R}^1$ satisfying $\|\gamma^{(2)}_{\mathrm{red}}\|_{TV} < \infty$. Further, suppose that $\mathbf{E}R_0 < \infty$ and $\Phi_0 \sim G$ with (not necessarily continuous) distribution function G. Then*

$$\varrho \int_{K^2} \int_{\mathbb{R}^2} w_{\varrho \mathbf{x}}(p_1) w_{\varrho \mathbf{y}}(p_2)\, \gamma^{(2)}(\mathrm{d}\mathbf{p}_{1,2})\, \mathrm{d}(\mathbf{x}, \mathbf{y}) \underset{\varrho\to\infty}{\longrightarrow} \lambda \big(\mathbf{E}|\Xi_0|_1\big)^2 \gamma^{(2)}_{\mathrm{red}}\big(\mathbb{R}^1\big) \int_{\mathbb{R}^1} \big(\mathbf{E}|g(p, \Phi_0) \cap K|_1\big)^2\, \mathrm{d}p.$$

Lemma 6. *If $\mathbf{E}R_0^2 < \infty$ and $\Phi_0 \sim G$ with (not necessarily continuous) distribution function G. Then*

$$J_\varrho(K) := \varrho \int_{K^2} \int_{\mathbb{R}^1} w^{\cap}_{\varrho \mathbf{x}, \varrho \mathbf{y}}(p)\, \mathrm{d}p\, \mathrm{d}(\mathbf{x}, \mathbf{y}) \underset{\varrho\to\infty}{\longrightarrow} 2\mathbf{E}|\Xi_0|_1^2 \int_0^{\pi} \int_0^{r_K(\varphi \pm \pi/2)} \left| K \cap \left(K + s v\left(\varphi \pm \frac{\pi}{2}\right)\right)\right|_2 \mathrm{d}s\, \mathrm{d}G(\varphi).$$

Proof of Lemma 4. We apply (3.11) for $k = m$ to $\alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m})$, which allows us to express the left-hand side of (4.1) as follows:

$$\sum_{\ell=1}^{m-1} \sum_{K_1 \cup \dots \cup K_\ell = \{1,\dots,m\}} \prod_{j=1}^{\ell} \int_{\mathbb{R}^{\# K_j}} \prod_{i \in K_j} w_{\varrho x_i}(p_i)\, \gamma^{(\# K_j)}\big(\mathrm{d}(p_i \colon i \in K_j)\big) + \lambda^m \int_{\mathbb{R}^m} \prod_{j=1}^{m} w_{\varrho x_j}(p_j)\, \mathrm{d}\mathbf{p}_{1,m}.$$

Applying Fubini's theorem shows that the last summand coincides with the right-hand side of (4.1). Hence the proof of Lemma 4 is complete if the remaining summands in the foregoing line disappear as $\varrho \to \infty$, and this follows by showing that, for $k = 2, \dots, m$,

$$\int\limits_{\mathbb{R}^k} \prod_{i=1}^{k} w_{\varrho x_i}(p_i)\, \gamma^{(k)}(\mathrm{d}\mathbf{p}_{1,k}) = \lambda \int\limits_{\mathbb{R}^k} w_{\varrho x_1}(p_1) \prod_{i=2}^{k} w_{\varrho x_i}(p_i + p_1)\, \gamma^{(k)}_{\mathrm{red}}(\mathrm{d}\mathbf{p}_{2,k})\, \mathrm{d}p_1$$

converges to 0 as $\varrho \to \infty$. Since $0 \leqslant w_{\varrho x_i}(p_i + p_1) \leqslant 1$ for $i = 3, \dots, k$, it suffices to prove that

$$\int\limits_{\mathbb{R}^{k-1}} \int\limits_{\mathbb{R}^1} \mathbf{P}\big(p_1 \in \Xi_0 + \varrho\langle v(\Phi_0), x_1\rangle\big) \mathbf{P}\big(p_1 \in \Xi_0 + \varrho\langle v(\Phi_0), x_2\rangle - p_2\big)\, \mathrm{d}p_1\, \big|\gamma^{(k)}_{\mathrm{red}}\big|(\mathrm{d}\mathbf{p}_{2,k}) \underset{\varrho\to\infty}{\longrightarrow} 0.$$

Since the total-variation measure $|\gamma^{(k)}_{\mathrm{red}}|$ is finite on $\mathbb{R}^{k-1}$ and the inner integral over $\mathbb{R}^1$ is bounded by $\mathbf{E}|\Xi_0|_1$, we have only to verify that the inner integral vanishes as $\varrho \to \infty$. For this purpose, we rewrite this integral as the expectation over indicator functions

$$\int\limits_{\mathbb{R}^1} \mathbf{E}\mathbf{1}_{\{\Xi_1 + \varrho\langle v(\Phi_1), x_1\rangle\}}(p_1) \mathbf{1}_{\{\Xi_2 + \varrho\langle v(\Phi_2), x_2\rangle - p_2\}}(p_1)\, \mathrm{d}p_1$$
$$= \mathbf{E}\big|\Xi_1 \cap \big(\Xi_2 - p_2 + \varrho\big(\big\langle v(\Phi_2), x_2\big\rangle - \big\langle v(\Phi_1), x_1\big\rangle\big)\big)\big|_1,$$

where $\Xi_i := [-R_i, R_i]$ and Φ_i for $i = 1, 2$ have the same distributions as $\Xi_0 = [-R_0, R_0]$ and Φ_0, respectively, and R_1, R_2, Φ_1, Φ_2 are mutually independent random variables. The right-hand expectation in the last line converges to 0 as $\varrho \to \infty$. This can be verified as follows: For $i = 1, 2$, fix two points $x_i = \|x_i\|(\cos(\alpha_i), \sin(\alpha_i)) \in \mathbb{R}^2$ and two points $v(\varphi_i) = (\cos(\varphi_i), \sin(\varphi_i))$ on the unit circle line. We easily see that the equality $\langle v(\varphi_1), x_1\rangle = \langle v(\varphi_2), x_2\rangle$ or, in other words, $\|x_1\| \cos(\varphi_1 - \alpha_1) = \|x_2\| \cos(\varphi_2 - \alpha_2)$ holds for finitely many pairs $\varphi_1, \varphi_2 \in [0, \pi]$. Hence, for two independent random angles Φ_1, Φ_2 with continuous distribution function G, we have $\mathbf{P}(\langle v(\Phi_1), x_1\rangle \neq \langle v(\Phi_2), x_2\rangle) = 1$ for any two points $x_1, x_2 \in \mathbb{R}^2$ with $\|x_1\| + \|x_2\| > 0$. □

Proof of Lemma 5. The stationarity of $\Psi \sim P$ implies $\gamma^{(2)}(\mathrm{d}\mathbf{p}_{1,2}) = \lambda\gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_2 - p_1)\, \mathrm{d}p_1$, so that

$$\varrho \int\limits_{\mathbb{R}^2} \int\limits_{K^2} w_{\varrho\mathbf{x}}(p_1) w_{\varrho\mathbf{y}}(p_2)\, \mathrm{d}(\mathbf{x}, \mathbf{y})\, \gamma^{(2)}(\mathrm{d}\mathbf{p}_{1,2}) = \varrho\, \lambda \int\limits_{\mathbb{R}^2} \int\limits_{K^2} w_{\varrho\mathbf{x}}(p_1) w_{\varrho\mathbf{y}}(p_2 + p_1)\, \mathrm{d}(\mathbf{x}, \mathbf{y})\, \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_2)\, \mathrm{d}p_1$$

with integration over points $\mathbf{x} = (x^{(1)}, x^{(2)}) \in K$ and $\mathbf{y} = (y^{(1)}, y^{(2)}) \in K$. As in the proof of Lemma 4, we express the product of the probabilities $w_{\varrho\mathbf{x}}(p_1) = \mathbf{P}(p_1 \in \{\cdots\})$ and $w_{\varrho\mathbf{y}}(p_2 + p_1) = \mathbf{P}(p_2 + p_1 \in \{\cdots\})$ as the expectation of the product of the corresponding indicator functions $\mathbf{1}_{\{\dots\}}(p_1)\, \mathbf{1}_{\{\dots\}}(p_2 + p_1)$. We fix $\Xi_i = \xi_i$ (compact sets in $\mathbb{R}^1$) and $\Phi_i = \varphi_i$ (angles in $[0, \pi]$) for $i = 1, 2$ and omit the expectation in front of all other integrals due to Fubini's theorem. The intensity λ is also suppressed. Thus we only treat the integral

$$J_\varrho(K, \xi_1, \varphi_1, \xi_2, \varphi_2) := \int\limits_{\mathbb{R}^2} \varrho \int\limits_{K^2} \mathbf{1}_{\xi_1 + \varrho\langle v(\varphi_1), \mathbf{x}\rangle}(p_1) \mathbf{1}_{\xi_2 + \varrho\langle v(\varphi_2), \mathbf{y}\rangle}(p_2 + p_1)\, \mathrm{d}(\mathbf{x}, \mathbf{y})\, \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_2)\, \mathrm{d}p_1. \tag{4.2}$$

Now we substitute $\mathbf{x} = \mathbf{s}O(-\varphi_1)$ and $\mathbf{y} = \mathbf{t}O(-\varphi_2)$, where $\mathbf{s} = (s^{(1)}, s^{(2)})$, $\mathbf{t} = (t^{(1)}, t^{(2)})$, and $O(\varphi)$ is defined by (3.4). Then $x^{(1)} = s^{(1)} \cos\varphi_1 - s^{(2)} \sin\varphi_1$, $x^{(2)} = s^{(1)} \sin\varphi_1 + s^{(2)} \cos\varphi_1$ and $y_1 = t^{(1)} \cos\varphi_2 - t^{(2)} \sin\varphi_2$, $y_2 = t^{(1)} \sin\varphi_2 + t^{(2)} \cos\varphi_2$. Hence, since $O(\varphi_i)^{-1} = O(-\varphi_i)$ for $i = 1, 2$, after a further substitution of p_1 by $p_1 + \varrho\, t^{(1)}$, we get that integral (4.2) takes the form

$$\varrho \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \mathbf{1}_{KO(\varphi_1)}(\mathbf{s}) \mathbf{1}_{KO(\varphi_2)}(\mathbf{t}) \mathbf{1}_{\xi_1 + \varrho(s^{(1)} - t^{(1)})}(p_1) \mathbf{1}_{\xi_2}(p_2 + p_1)\, \mathrm{d}\mathbf{s}\, \mathrm{d}\mathbf{t}\, \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_2)\, \mathrm{d}p_1.$$

The invariance properties of the one-dimensional Hausdorff measure on $\mathbb{R}^2$ (also denoted by $|\cdot|_1$) yield $\int_{\mathbb{R}^1} \mathbf{1}_{KO(\varphi_1)}(\mathbf{s})\, \mathrm{d}s^{(2)} = |g(s^{(1)}, \varphi_1) \cap K|_1$ and $\int_{\mathbb{R}^1} \mathbf{1}_{KO(\varphi_2)}(\mathbf{t})\, \mathrm{d}t^{(2)} = |g(t^{(1)}, \varphi_2) \cap K|_1$. In the resulting integral, we substitute $s^{(1)} = s/\varrho + t^{(1)}$ and $t^{(1)} = t$, so that $J_\varrho(K, \xi_1, \varphi_1, \xi_2, \varphi_2)$ is equal to

$$\begin{aligned} &\int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^1} \int\limits_{\mathbb{R}^1} \Big| g\Big(\frac{s}{\varrho} + t,\, \varphi_1\Big) \cap K \Big|_1 \big|g(t, \varphi_2) \cap K\big|_1 \mathbf{1}_{\xi_1 + s}(p_1) \mathbf{1}_{\xi_2}(p_2 + p_1)\, \mathrm{d}s\, \mathrm{d}t\, \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_2)\, \mathrm{d}p_1 \\ &\quad \underset{\varrho \to \infty}{\longrightarrow} |\xi_1|_1 |\xi_2|_1 \gamma^{(2)}_{\mathrm{red}}(\mathbb{R}^1) \int\limits_{\mathbb{R}^1} \big|g(t, \varphi_1) \cap K\big|_1 \big|g(t, \varphi_2) \cap K\big|_1\, \mathrm{d}t. \end{aligned} \tag{4.3}$$

To justify the latter limit, we have used that $|g(s/\varrho + t,\, \varphi_1) \cap K|_1 \leqslant \mathrm{diam}(K)$, so that Lebesgue's dominated convergence theorem can be applied. Furthermore, we easily see that

$$\big|J_\varrho(K, \xi_1, \varphi_1, \xi_2, \varphi_2)\big| \leqslant \mathrm{diam}(K) |K|_2 |\xi_1|_1 |\xi_2|_1 \big\|\gamma^{(2)}_{\mathrm{red}}\big\|_{TV}. \tag{4.4}$$

Combining (4.3) with the independence assumptions shows that the limit of $\lambda \mathbf{E} J_\varrho(K, \Xi_1, \Phi_1, \Xi_2, \Phi_2)$ as $\varrho \to \infty$ coincides with that stated in Lemma 5. □

Proof of Lemma 6. We rewrite the integral $J_\varrho(K)$ as follows:

$$\begin{aligned} J_\varrho(K) &= \varrho \int\limits_{\mathbb{R}^2} \mathbf{1}_{K \oplus (-K)}(\mathbf{y}) \big|K \cap (K - \mathbf{y})\big|_2 \mathbf{E} \big|\Xi_0 \cap \big(\Xi_0 + \varrho \langle v(\Phi_0), \mathbf{y} \rangle\big)\big|_1\, \mathrm{d}\mathbf{y} \\ &= \varrho \int\limits_0^{2\pi} \int\limits_0^{\infty} \mathbf{1}_{K \oplus (-K)}\big(r\, v(\psi)\big) \big|K \cap \big(K - r\, v(\psi)\big)\big|_2 \mathbf{E} \big|\Xi_0 \cap \big(\Xi_0 + \varrho r \cos(\psi - \Phi_0)\big)\big|_1 r\, \mathrm{d}r\, \mathrm{d}\psi \\ &= \varrho \mathbf{E} \int\limits_0^{2\pi} \int\limits_0^{r_K(\psi + \Phi_0)} \big|K \cap \big(K - r\, v(\Phi_0 + \psi)\big)\big|_2 \big|\Xi_0 \cap \big(\Xi_0 + \varrho r \cos(\psi)\big)\big|_1 r\, \mathrm{d}r\, \mathrm{d}\psi \end{aligned}$$

with $r_K(\psi)$ as defined in Theorem 2. Here we have used the substitution $\mathbf{y} = r\, v(\psi)$ and the independence of Φ_0 and R_0. Further, since $v(\psi + \pi) = -v(\psi)$, by the shift-invariance of $|\cdot|_1$, the motion-invariance of $|\cdot|_2$, and the fact that by the definition of $r_K(\psi)$, $r > r_K(\psi)$ iff $r\, v(\psi) \notin K \oplus (-K)$ iff $K \cap (K + r\, v(\psi)) = \emptyset$, we arrive at

$$\begin{aligned} J_\varrho(K) &= 2\varrho \mathbf{E} \int\limits_0^{\pi} \int\limits_0^{\infty} \big|K \cap \big(K + r\, v(\Phi_0 + \psi)\big)\big|_2 \big|\Xi_0 \cap \big(\Xi_0 + \varrho r \cos(\psi)\big)\big|_1 r\, \mathrm{d}r\, \mathrm{d}\psi \\ &= 2\mathbf{E} \int\limits_0^{\infty} \int\limits_{-r}^{r} \Big|K \cap \Big(K + r\, v\Big(\Phi_0 + \arccos\frac{z}{r}\Big)\Big)\Big|_2 \big|\Xi_0 \cap (\Xi_0 + \varrho z)\big|_1 \frac{\varrho\, \mathrm{d}z\, r\, \mathrm{d}r}{\sqrt{r^2 - z^2}} \end{aligned}$$

by substituting $\psi = \arccos(z/r)$ for $z \in [-r, r]$ and changing the order of integration. After changing once more the integration order over z and r and substituting $z = u/\varrho$, we can proceed with the abbreviation

$$h(r,z,\varphi) := r\,v\biggl(\varphi + \arccos\frac{z}{r}\biggr) = \bigl(z\cos\varphi - \sqrt{r^2 - z^2}\sin\varphi,\ z\sin\varphi + \sqrt{r^2 - z^2}\cos\varphi\bigr),$$

where $0 \leqslant \|h(r,z,\varphi)\| = r \leqslant r_K := \max\{r_K(\psi)\colon\ 0 \leqslant \psi \leqslant \pi\} \leqslant \operatorname{diam}(K)$, which shows that

$$\begin{aligned} J_\varrho(K) &= 2\mathbf{E}\int_{\mathbb{R}^1}\int_{|u|/\varrho}^{r_K}\biggl|K \cap \biggl(K + h\biggl(r, \frac{|u|}{\varrho}, \varPhi_0\biggr)\biggr)\biggr|_2 \bigl|\varXi_0 \cap (\varXi_0 + u)\bigr|_1 \frac{r\,\mathrm{d}r\,\mathrm{d}u}{\sqrt{r^2 - \frac{u^2}{\varrho^2}}} \\ &\underset{\varrho\to\infty}{\longrightarrow} 2\mathbf{E}\int_{\mathbb{R}^1}\int_0^{r_K}\biggl|K \cap \biggl(K + r\,v\biggl(\varPhi_0 + \frac{\pi}{2}\biggr)\biggr)\biggr|_2 \mathbf{E}\bigl|\varXi_0 \cap (\varXi_0 + u)\bigr|_1\,\mathrm{d}r\,\mathrm{d}u. \end{aligned} \tag{4.5}$$

In the last step, we could apply Lebesgue's dominated convergence theorem since the inner integral in (4.5) over r is bounded by $|K|_2\ \operatorname{diam}(K)$ and the mapping $z \mapsto h(r,z,\varphi)$ is continuous in $z = 0$ with $h(r,0,\varphi) = r\,v(\varphi + \pi/2)$ and $\arccos(0) = \pi/2$. Finally, we use the relation $\int_{\mathbb{R}^1} |\varXi_0 \cap (\varXi_0 + u)|_1\,\mathrm{d}u = |\varXi_0|_1^2 = 4R_0^2$ and the independence of $\varPhi_0$ and $\varXi_0$, which provide the statement of Lemma 6. □

5 Proofs of Theorems 1 and 2

Proof of Theorem 1. According to the definition of $L^2(\mathbf{P})$-convergence, the limit (2.2) is proved if

$$\mathbf{E}\biggl(\frac{|\varXi \cap \varrho K|_2}{|\varrho K|_2} - \bigl(1 - \exp\bigl\{-\lambda\mathbf{E}|\varXi_0|_1\bigr\}\bigr)\biggr)^2 = \frac{\operatorname{Var}|\varXi \cap \varrho K|_2}{|\varrho K|_2^2} + \biggl(\frac{\mathbf{E}|\varXi \cap \varrho K|_2}{|\varrho K|_2} - \bigl(1 - \exp\bigl\{-\lambda\mathbf{E}|\varXi_0|_1\bigr\}\bigr)\biggr)^2$$

vanishes as $\varrho \to \infty$. In view of Corollary 2, it remains to prove that $\varrho^{-4}\ \operatorname{Var}|\varXi \cap \varrho K|_2 \to 0$ as $\varrho \to \infty$. For this purpose, we use the variance formula (3.8) of Lemma 2 and show that

$$G_P\bigl[1 - w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(\cdot)\bigr] - G_P\bigl[1 - w_{\varrho\mathbf{x}}(\cdot)\bigr]G_P\bigl[1 - w_{\varrho\mathbf{y}}(\cdot)\bigr] \underset{\varrho\to\infty}{\longrightarrow} 0 \tag{5.1}$$

for any distinct points $\mathbf{x}, \mathbf{y} \in K \setminus \{\mathbf{o}\}$.

For this, we make use of the finite expansion (3.10) of the PGF $G_P[1 - w_{\varrho\mathbf{x}}(\cdot)]$ with remainder term, where $w_{\varrho\mathbf{x}}$ can be replaced by any Borel-measurable function $w : \mathbb{R}^1 \mapsto [0,1]$. For brevity, we put

$$S_m(w) := 1 + \sum_{k=1}^{m-1}(-1)^k T_k(w) \quad \text{with } T_k(w) := \frac{1}{k!}\int_{\mathbb{R}^k}\prod_{j=1}^{k} w(p_j)\,\alpha^{(k)}(\mathrm{d}\mathbf{p}_{1,k}) \text{ for } 1 \leqslant k \leqslant m \in \mathbb{N}.$$

Hence (3.10) reads as $|G_P[1 - w(\cdot)] - S_m(w)| \leqslant T_m(w)$, which leads us to the following estimate:

$$\begin{aligned} &\bigl|G_P\bigl[1 - w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(\cdot)\bigr] - G_P\bigl[1 - w_{\varrho\mathbf{x}}(\cdot)\bigr]G_P\bigl[1 - w_{\varrho\mathbf{y}}(\cdot)\bigr] - \bigl(S_m\bigl(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}\bigr) - S_m(w_{\varrho\mathbf{x}})\,S_m(w_{\varrho\mathbf{y}})\bigr)\bigr| \\ &\quad\leqslant T_m\bigl(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}\bigr) + T_m(w_{\varrho\mathbf{x}}) + T_m(w_{\varrho\mathbf{y}}) + T_m(w_{\varrho\mathbf{x}})T_m(w_{\varrho\mathbf{y}}) \quad \text{for } \mathbf{x}, \mathbf{y} \in K \setminus \{\mathbf{o}\} \text{ and } m \geqslant 2, \end{aligned} \tag{5.2}$$

where we have additionally used that $G_P[1 - w(\cdot)] \leqslant 1$ and $S_m(w) \leqslant G_P[1 - w(\cdot)] + T_m(w)$. We are now in a position to apply the limit (4.1) of Lemma 4, which yields that for any $\mathbf{x} \in K \setminus \{\mathbf{o}\}$ and $m \in \mathbb{N}$,

$$T_m(w_{\varrho\mathbf{x}}) \underset{\varrho\to\infty}{\longrightarrow} \frac{(\lambda \mathbf{E}|\Xi_0|_1)^m}{m!} \quad \text{and} \quad S_m(w_{\varrho\mathbf{x}}) \underset{\varrho\to\infty}{\longrightarrow} \sum_{k=0}^{m-1} \frac{(-\lambda \mathbf{E}|\Xi_0|_1)^k}{k!} = \mathrm{e}^{-\lambda \mathbf{E}|\Xi_0|_1} + \theta_1 \frac{(\lambda \mathbf{E}|\Xi_0|_1)^m}{m!}$$

for some $\theta_1 \in [-1, 1]$ in accordance with $|\mathrm{e}^{-x} - \sum_{k=0}^{m-1}(-x)^k/k!| \leqslant x^m/m!$ for all $m \in \mathbb{N}$ and $x \geqslant 0$.

Next, we have to find the limit of $T_m(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}})$ as $\varrho \to \infty$. Using the relation $w^{\cup}_{\mathbf{x},\mathbf{y}}(p) = w_{\mathbf{x}}(p) + w_{\mathbf{y}}(p) - w^{\cap}_{\mathbf{x},\mathbf{y}}(p)$ and taking into account that the factorial moment measure $\alpha^{(m)}$ is invariant under permutations of its m components, we may write

$$T_m\big(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}\big) = \frac{1}{m!} \int\limits_{\mathbb{R}^m} \prod_{j=1}^{m} \big(w_{\varrho\mathbf{x}}(p_j) + w_{\varrho\mathbf{y}}(p_j)\big)\, \alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m}) \tag{5.3}$$

$$+ \frac{1}{m!} \sum_{\ell=1}^{m} \binom{m}{\ell} \int\limits_{\mathbb{R}^m} \prod_{i=1}^{\ell} w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_i) \prod_{j=\ell+1}^{m} \big(w_{\varrho\mathbf{x}}(p_j) + w_{\varrho\mathbf{y}}(p_j)\big)\, \alpha^{(m)}(\mathrm{d}\mathbf{p}_{1,m}).$$

There is at least one term $w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_i) = \mathbf{P}(p_i \in (\Xi_0 + \varrho\langle v(\Phi_0), \mathbf{x}\rangle) \cap (\Xi_0 + \varrho\langle v(\Phi_0), \mathbf{y}\rangle))$ in each summand of the last line that will be integrated over $\mathbb{R}^1$ with respect to $\mathrm{d}p_i$, so that after expressing $\alpha^{(m)}$ by cumulant measures (see (3.11)), the expectation $\mathbf{E}|\Xi_0 \cap (\Xi_0 + \varrho\,\langle v(\Phi_0),\, \mathbf{y} - \mathbf{x}\rangle)|_1$ emerges and vanishes as $\varrho \to \infty$ if $\mathbf{x} \neq \mathbf{y}$. Thus the last line completely vanishes as $\varrho \to \infty$, whereas the line (5.3) converges to the limit $(2\lambda \mathbf{E}|\Xi_0|_1)^m/m!$ as $\varrho \to \infty$ by applying the limit (4.1) once more. Therefore for any $m \in \mathbb{N}$ and $\mathbf{x} \neq \mathbf{y}$, we obtain that $T_m(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}) \to (2\lambda \mathbf{E}|\Xi_0|_1)^m/m!$ as $\varrho \to \infty$ and

$$S_m\big(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}\big) \underset{\varrho\to\infty}{\longrightarrow} \sum_{k=0}^{m-1} \frac{(-2\,\lambda \mathbf{E}|\Xi_0|_1)^k}{k!} = \mathrm{e}^{-2\lambda \mathbf{E}|\Xi_0|_1} + \theta_2 \frac{(2\lambda \mathbf{E}|\Xi_0|_1)^m}{m!}$$

for some $\theta_2 \in [-1, 1]$. The last limit, combined with the above limits of $S_m(w_{\varrho\mathbf{x}})$ and $S_m(w_{\varrho\mathbf{y}})$, leads to

$$\overline{\lim_{\varrho\to\infty}} \big|S_m\big(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}\big) - S_m(w_{\varrho\mathbf{x}}) S_m(w_{\varrho\mathbf{y}})\big| \leqslant \frac{(2\lambda \mathbf{E}|\Xi_0|_1)^m}{m!} + 2\frac{(\lambda \mathbf{E}|\Xi_0|_1)^m}{m!} + \frac{(\lambda \mathbf{E}|\Xi_0|_1)^{2m}}{(m!)^2}.$$

For a given $\varepsilon \in (0, 1]$, we find a large enough $m(\varepsilon)$ such that $(2\lambda \mathbf{E}|\Xi_0|_1)^m/m! \leqslant \varepsilon$ for all $m \geqslant m(\varepsilon)$. Thus the right-hand side of the last inequality does not exceed $2\,\varepsilon + \varepsilon^2$ for sufficiently large m. The same bound can be obtained for the limit (as $\varrho \to \infty$) of the four terms in line (5.2). Finally, after summarizing all ε-bounds of the above limiting terms, we arrive at

$$\overline{\lim_{\varrho\to\infty}} \big|G_P\big[1 - w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(\cdot)\big] - G_P\big[1 - w_{\varrho\mathbf{x}}(\cdot)\big] G_P\big[1 - w_{\varrho\mathbf{y}}(\cdot)\big]\big| \leqslant 2\big(2\varepsilon + \varepsilon^2\big) \leqslant 6\varepsilon. \tag{5.4}$$

This, together with (5.3), implies $\varrho^{-4}\,\mathrm{Var}\,|\Xi \cap \varrho K|_2 \to 0$ as $\varrho \to \infty$, completing the proof of Theorem 1. □

Proof of Theorem 2. Lemma 2 yields the equality

$$\varrho^{-3}\,\mathrm{Var}\,|\Xi \cap \varrho K|_2 = \int\limits_{K^2} \varrho\big(G_P\big[1 - w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}(\cdot)}\big] - G_P\big[1 - w_{\varrho\mathbf{x}}(\cdot)\big] G_P\big[1 - w_{\varrho\mathbf{y}}(\cdot)\big]\big)\, \mathrm{d}(\mathbf{x}, \mathbf{y}). \tag{5.5}$$

We first rewrite the integrand on the right-hand side of (5.5) as follows:

$$\begin{aligned}\varrho\big(G_P\big[1-w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(\cdot)\big]-G_P\big[1-w_{\varrho\mathbf{x}}(\cdot)\big]G_P\big[1-w_{\varrho\mathbf{y}}(\cdot)\big]\big)\\ =\varrho\big(\exp\big\{L(\varrho\mathbf{x},\varrho\mathbf{y})\big\}-1\big)G_P\big[1-w_{\mathbf{x}}(\cdot)\big]G_P\big[1-w_{\mathbf{y}}(\cdot)\big],\end{aligned}\tag{5.6}$$

where $L(\mathbf{x},\mathbf{y}) := \log G_P[1-w^{\cup}_{\mathbf{x},\mathbf{y}}(\cdot)] - \log G_P[1-w_{\mathbf{x}}(\cdot)] - \log G_P[1-w_{\mathbf{y}}(\cdot)]$.

To study the behavior of $L(\varrho\mathbf{x},\varrho\mathbf{y})$ (as $\varrho\to\infty$), we use an expansion of $\log G_P[1-w(\cdot)]$ in terms of the factorial cumulant measures $\gamma^{(k)}$ of $\Psi\sim P$ (see (2.1)):

$$\log G_P\big[1-w(\cdot)\big]=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int\limits_{\mathbb{R}^k}\prod_{j=1}^{n}w(p_j)\,\gamma^{(n)}(\mathrm{d}\mathbf{p}_{1,n})\quad\text{(see [2, p. 146])},\tag{5.7}$$

provided the sum in (5.7) is convergent. In view of (5.5) and the inequality $|\mathrm{e}^x-1-x|\leqslant x^2\mathrm{e}^{\max(x,0)}/2$, we have to find a uniform bound of $L(\varrho\mathbf{x},\varrho\mathbf{y})$ and to calculate the limits

$$\lim_{\varrho\to\infty}\varrho\int\limits_{K^2}L(\varrho\mathbf{x},\varrho\mathbf{y})\,\mathrm{d}(\mathbf{x},\mathbf{y})\quad\text{and}\quad\lim_{\varrho\to\infty}\varrho\int\limits_{K^2}\big(L(\varrho\mathbf{x},\varrho\mathbf{y})\big)^2\,\mathrm{d}(\mathbf{x},\mathbf{y}).\tag{5.8}$$

We start by noting that relations (3.10), (4.1), and (5.4) under the assumptions of Theorem 1 imply that

$$\lim_{\varrho\to\infty}G_P\big[1-w_{\varrho\mathbf{x}}(\cdot)\big]=\mathrm{e}^{-\lambda\mathbf{E}|\Xi_0|_1},\qquad\lim_{\varrho\to\infty}G_P\big[1-w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(\cdot)\big]=\mathrm{e}^{-2\lambda\mathbf{E}|\Xi_0|_1},\tag{5.9}$$

and

$$\lim_{\varrho\to\infty}L(\varrho\mathbf{x},\varrho\mathbf{y})=0\tag{5.10}$$

for all distinct points $\mathbf{x},\mathbf{y}\in K\setminus\{\mathbf{o}\}$. The limit (5.10) suggests that

$$\lim_{\varrho\to\infty}\varrho\int\limits_{K^2}\big(\exp\big\{L(\varrho\mathbf{x},\varrho\mathbf{y})\big\}-1\big)\,\mathrm{d}(\mathbf{x},\mathbf{y})=\lim_{\varrho\to\infty}\varrho\int\limits_{K^2}L(\varrho\mathbf{x},\varrho\mathbf{y})\,\mathrm{d}(\mathbf{x},\mathbf{y}),\tag{5.11}$$

where the existence of the limit (5.11) has yet to be shown. A rigorous proof that the second limit in (5.8) vanishes as $\varrho\to\infty$ together with a uniform bound of $L(\varrho\mathbf{x},\varrho\mathbf{y})$ will be given after calculation of the first limit in (5.8). By combining (5.5) and (5.6) with limits (5.8)–(5.10) and (5.11) we see that the limit of $\varrho^{-3}\operatorname{Var}|\Xi\cap\varrho K|_2$ as $\varrho\to\infty$ (if it exists) coincides with

$$\mathrm{e}^{-2\lambda\mathbf{E}|\Xi_0|_1}\lim_{\varrho\to\infty}\int\limits_{K^2}\varrho L(\varrho\mathbf{x},\varrho\mathbf{y})\,\mathrm{d}(\mathbf{x},\mathbf{y}).$$

By using expansion (5.7) we are able to express the double integral of (5.8) as follows:

$$\int\limits_{K^2}\varrho\,L(\varrho\mathbf{x},\varrho\mathbf{y})\,\mathrm{d}(\mathbf{x},\mathbf{y})=\sum_{n\geqslant1}\frac{(-1)^nT_n^{(\varrho)}(K)}{n!}\quad\text{with }T_n^{(\varrho)}(K):=\varrho\int\limits_{K^2}T_n(\varrho\mathbf{x},\varrho\mathbf{y})\,\mathrm{d}(\mathbf{x},\mathbf{y}),$$

where the integrands $T_n(\varrho\mathbf{x},\varrho\mathbf{y})$ for $n\in\mathbb{N}$ are defined by

$$T_n(\varrho\mathbf{x},\varrho\mathbf{y}):=\int\limits_{\mathbb{R}^n}\bigg(\prod_{j=1}^{n}w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_j)-\prod_{j=1}^{n}w_{\varrho\mathbf{x}}(p_j)-\prod_{j=1}^{n}w_{\varrho\mathbf{y}}(p_j)\bigg)\gamma^{(n)}(\mathrm{d}\mathbf{p}_{1,n}).\tag{5.12}$$

Since $\gamma^{(1)}(\mathrm{d}p) = \lambda\,\mathrm{d}p$ and $w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p) - w_{\varrho\mathbf{x}}(p) - w_{\varrho\mathbf{y}}(p) = -w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p)$, we get

$$-T_1^{(\varrho)}(K) = \lambda \int\limits_{K^2}\int\limits_{\mathbb{R}^1} \varrho w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p)\,\mathrm{d}p\,\mathrm{d}(\mathbf{x},\mathbf{y}) = \lambda J_\varrho(K) \underset{\varrho\to\infty}{\longrightarrow} 2\lambda\mathbf{E}|\Xi_0|_1^2 C_2^{G,K},$$

where the right-hand limit is just the statement of Lemma 6. The proof of Lemma 6 reveals that $|T_1^{(\varrho)}(K)| \leqslant \lambda J_\varrho(K) \leqslant 2\lambda\mathbf{E}|\Xi_0|_1^2|K|_2\operatorname{diam}(K)$. We easily see that $|T_1(\varrho\mathbf{x},\varrho\mathbf{y})| \leqslant \lambda\mathbf{E}|\Xi_0|_1$ and $T_1(\varrho\mathbf{x},\varrho\mathbf{y}) = -\lambda\mathbf{E}|\Xi_0 \cap (\Xi_0 + \varrho\langle v(\Phi_0),\,\mathbf{y}-\mathbf{x}\rangle)|_1$ vanishes as $\varrho\to\infty$ for $\mathbf{x}\neq\mathbf{y}$. In the next step, we derive a uniform bound of $T_2^{(\varrho)}(K)$ and determine its limit as $\varrho\to\infty$.

Since the integrand in (5.12) for $n=2$ is symmetric in $\mathbf{x}$, $\mathbf{y}$ and in p_1, p_2, it follows that

$$T_2^{(\varrho)}(K) = 2\varrho \int\limits_{K^2}\int\limits_{\mathbb{R}^2} w_{\varrho\mathbf{x}}(p_1) w_{\varrho\mathbf{y}}(p_2)\,\gamma^{(2)}(\mathrm{d}\mathbf{p}_{1,2})\,\mathrm{d}(\mathbf{x},\mathbf{y}) + \widetilde{T}_2^{(\varrho)}(K), \tag{5.13}$$

where, in view of $|\gamma^{(2)}_{\mathrm{red}}|(\mathbb{R}^1) < \infty$ and the dominated convergence theorem,

$$\begin{aligned}\big|\widetilde{T}_2^{(\varrho)}(K)\big| &= \varrho\bigg|\int\limits_{K^2}\int\limits_{\mathbb{R}^2} w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_1)\big(w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_2) + w_{\varrho\mathbf{x}}(p_2) + w_{\varrho\mathbf{y}}(p_2)\big)\,\gamma^{(2)}(\mathrm{d}\mathbf{p}_{1,2})\,\mathrm{d}(\mathbf{x},\mathbf{y})\bigg| \\ &\leqslant 4\lambda\varrho \int\limits_{K^2}\int\limits_{\mathbb{R}^1} w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p)\mathbf{E}\big|\gamma^{(2)}_{\mathrm{red}}\big|\big(\Xi_0 + \varrho\,\langle v(\Phi_0),\mathbf{x}\rangle - p\big)\,\mathrm{d}p\,\mathrm{d}(\mathbf{x},\mathbf{y}) \underset{\varrho\to\infty}{\longrightarrow} 0.\end{aligned} \tag{5.14}$$

In the last line, we have used the same arguments as in the proof of Lemma 6, among them, the uniform estimate $J_\varrho(K) \leqslant 2\mathbf{E}|\Xi_0|_1^2|K|_2\operatorname{diam}(K)$. Finally, Lemma 5 and (5.13) show that

$$\frac{T_2^{(\varrho)}(K)}{2} \underset{\varrho\to\infty}{\longrightarrow} \lambda\big(\mathbf{E}|\Xi_0|_1\big)^2\gamma^{(2)}_{\mathrm{red}}(\mathbb{R}^1)\int\limits_{\mathbb{R}^1}\big(\mathbf{E}\big|g(p,\Phi_0)\cap K\big|_1\big)^2\,\mathrm{d}p = \lambda\big(\mathbf{E}|\Xi_0|_1\big)^2\,\gamma^{(2)}_{\mathrm{red}}(\mathbb{R}^1)C_1^{G,K}.$$

In addition, we can derive a uniform bound of $T_2^{(\varrho)}(K)$. From (5.14) and the above bound of $T_1^{(\varrho)}(K)$ we get $|\widetilde{T}_2^{(\varrho)}(K)| \leqslant 4\|\gamma^{(2)}_{\mathrm{red}}\|_{TV}|T_1^{(\varrho)}(K)| \leqslant 8\lambda|K|_2\operatorname{diam}(K)\|\gamma^{(2)}_{\mathrm{red}}\|_{TV}\mathbf{E}|\Xi_0|_1^2$. Hence we see from (4.4) and (5.12) that for two independent pairs (Ξ_i,Φ_i), $i=1,2$, with the same distribution as (Ξ_0,Φ_0), we have the following estimate:

$$\big|T_2^{(\varrho)}(K)\big| \leqslant 2\lambda\mathbf{E}\big|J_\varrho(K,\Xi_1,\Phi_1,\Xi_2,\Phi_2)\big| + \big|\widetilde{T}_2^{(\varrho)}(K)\big| \leqslant 10\lambda|K|_2\operatorname{diam}(K)\mathbf{E}|\Xi_0|_1^2\big\|\gamma^{(2)}_{\mathrm{red}}\big\|_{TV}.$$

Obviously, the limit (2.3) coincides with $\lim_{\varrho\to\infty}(-T_1^{(\varrho)}(K) + T_2^{(\varrho)}(K)/2)$. Thus the existence of the first limit (5.8) is proved. To accomplish the proof of Theorem 2, we next show that

$$\lim_{\varrho\to\infty} T_n^{(\varrho)}(K) = 0 \quad\text{and}\quad \sup_{\varrho\geqslant 1}\frac{|T_n^{(\varrho)}(K)|}{n!} \leqslant C_n^K \quad \text{for } n\geqslant 3 \text{ such that } \sum_{n\geqslant 3} C_n^K < \infty. \tag{5.15}$$

For this purpose, we need suitable uniform (w.r.t. $\varrho(\geqslant 1)$) upper bounds of the integrals $T_n^{(\varrho)}(K)$ in (5.12), which vanish for $n\geqslant 3$ as $\varrho\to\infty$. First, we derive upper bounds of integrals (5.12) for $n\in\mathbb{N}$. Using the reduced factorial cumulant measures $\gamma^{(n)}_{\mathrm{red}}$ defined (in differential notation) by $\gamma^{(n)}(\mathrm{d}\mathbf{p}_{1,n}) = \lambda\,\gamma^{(n)}_{\mathrm{red}}((\mathrm{d}p_i - p_j\colon i\neq j))\,\mathrm{d}p_j$ for $j=1,\dots,n$ and the boundedness of the total-variation measure $|\gamma^{(n)}_{\mathrm{red}}|$ on $\mathbb{R}^{n-1}$, after some elementary calculations, we obtain the following estimates:

$$\begin{aligned} &T_{n,1}(\varrho\mathbf{x},\varrho\mathbf{y}) \\ &\quad := \left| \int\limits_{\mathbb{R}^n} \left(\prod_{i=1}^n \bigl(w_{\varrho\mathbf{x}}(p_i)+w_{\varrho\mathbf{y}}(p_i)\bigr) - \prod_{i=1}^n w_{\varrho\mathbf{x}}(p_i) - \prod_{i=1}^n w_{\varrho\mathbf{y}}(p_i) \right) \gamma^{(n)}(\mathrm{d}\mathbf{p}_{1,n}) \right| \\ &\quad \leqslant \lambda \sum_{k=1}^{n-1} \binom{n}{k} \int\limits_{\mathbb{R}^1} w_{\varrho\mathbf{x}}(p) \int\limits_{\mathbb{R}^{n-1}} \prod_{i=2}^k w_{\varrho\mathbf{x}}(p_i+p) \prod_{j=k+1}^n w_{\varrho\mathbf{y}}(p_j+p) \, \bigl|\gamma_{\mathrm{red}}^{(n)}\bigr|(\mathrm{d}\mathbf{p}_{2,n})\, \mathrm{d}p \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} &T_{n,2}(\mathbf{x},\mathbf{y}) \\ &\quad := \left| \int\limits_{\mathbb{R}^n} \left(\prod_{i=1}^n \bigl(w_{\varrho\mathbf{x}}(p_i)+w_{\varrho\mathbf{y}}(p_i)\bigr) - \prod_{i=1}^n w^{\cup}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_i) \right) \gamma^{(n)}(\mathrm{d}\mathbf{p}_{1,n}) \right| \\ &\quad \leqslant \lambda \sum_{k=1}^{n} k \binom{n}{k} \int\limits_{\mathbb{R}^1} w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p) \int\limits_{\mathbb{R}^{n-1}} \prod_{i=2}^k w_{\varrho\mathbf{x}}(p_i+p) \prod_{j=k+1}^n w_{\varrho\mathbf{y}}(p_j+p) \, \bigl|\gamma_{\mathrm{red}}^{(n)}\bigr|(\mathrm{d}\mathbf{p}_{2,n})\, \mathrm{d}p. \end{aligned} \tag{5.17}$$

For a Brillinger-mixing PP $\Psi \sim P$, it is easy to show that $T_{n,1}(\varrho\mathbf{x},\varrho\mathbf{y}) \leqslant \lambda \mathbf{E}|\Xi_0|_1 \|\gamma_{\mathrm{red}}^{(n)}\|_{TV}(2^n-2)$ and $T_{n,2}(\varrho\mathbf{x},\varrho\mathbf{y}) \leqslant \lambda \mathbf{E}|\Xi_0|_1 \|\gamma_{\mathrm{red}}^{(n)}\|_{TV} n2^n$ for $n \geqslant 1$. If $\Psi \sim P$ is strongly Brillinger mixing, that is, $\|\gamma_{\mathrm{red}}^{(n)}\|_{TV} \leqslant ab^n n!$ for $n \geqslant 1$ (see Definition 2), then we get that $|T_n(\varrho\mathbf{x},\varrho\mathbf{y})| \leqslant T_{n,1}(\varrho\mathbf{x},\varrho\mathbf{y}) + T_{n,2}(\varrho\mathbf{x},\varrho\mathbf{y}) \leqslant \lambda \mathbf{E}|\Xi_0|_1 \times (n+1)a(2b)^n n!$. Thus we obtain a uniform estimate of $L(\varrho\mathbf{x},\varrho\mathbf{y})$:

$$\bigl|L(\varrho\mathbf{x},\varrho\mathbf{y})\bigr| \leqslant \sum_{n=1}^{\infty} \frac{|T_n(\varrho\mathbf{x},\varrho\mathbf{y})|}{n!} \leqslant \frac{4\lambda(1-b)}{(1-2b)^2} \mathbf{E}|\Xi_0|_1. \tag{5.18}$$

Similar uniform bounds of $L(\varrho\mathbf{x},\varrho\mathbf{y})$ can be shown if $\Psi \sim P$ is strongly L_q-Brillinger mixing (resp., strongly L_q^*-Brillinger mixing). The derivation of these bounds is completely analogous to the proof of estimate (5.22) (resp., (5.24)) below. The details are left to the reader.

Next, we prove the announced relations in (5.15). Obviously, $|T_n^{(\varrho)}(K)| \leqslant T_{n,1}^{(\varrho)}(K) + T_{n,2}^{(\varrho)}(K)$ for $n \in \mathbb{N}$, where

$$T_{n,i}^{(\varrho)}(K) = \int\limits_{K^2} \varrho\, T_{n,i}(\varrho\mathbf{x},\varrho\mathbf{y})\, \mathrm{d}(\mathbf{x},\mathbf{y}) \quad \text{for } i=1,2.$$

First, we consider the integrals $T_{n,1}^{(\varrho)}(K)$ for $n \geqslant 3$. The bound of the integral $T_{n,1}(\varrho\mathbf{x},\varrho\mathbf{y})$ in (5.16) shows that $T_{n,1}^{(\varrho)}(K) \leqslant \lambda \sum_{k=1}^{n-1} \binom{n}{k} I_{n,k}^{(\varrho)}(K)$, where

$$I_{n,k}^{(\varrho)}(K) := \int\limits_{K^2} \int\limits_{\mathbb{R}^1} \varrho\, w_{\varrho\mathbf{x}}(p) \int\limits_{\mathbb{R}^{n-1}} \prod_{i=2}^k w_{\varrho\mathbf{x}}(p_i+p) \prod_{j=k+1}^n w_{\varrho\mathbf{y}}(p_j+p) \, \bigl|\gamma_{\mathrm{red}}^{(n)}\bigr|(\mathrm{d}\mathbf{p}_{2,n})\, \mathrm{d}p\, \mathrm{d}(\mathbf{x},\mathbf{y})$$

for $k = 2,\dots,n-1$. As in (4.2), we substitute $\mathbf{x} = \mathbf{u}\,O(-\Phi_1)$ and $\mathbf{y} = \mathbf{v}\,O(-\Phi_n)$ with $O(\varphi)$ defined by (3.4). Since $O^{-1}(\varphi) = O(-\varphi)$ and $\det(O(\varphi)) = 1$, it follows that $\mathbf{u} = \mathbf{x}\,O(\Phi_1)$, $\mathbf{v} = \mathbf{y}\,O(\Phi_n)$, and $\langle v(\Phi_i),\mathbf{x}\rangle = \langle v(\Phi_i),\mathbf{u}\,O(-\Phi_1)\rangle = \langle v(\Phi_i - \Phi_1),\mathbf{u}\rangle$ for $i = 1,\dots,k$ and $\langle v(\Phi_j),\mathbf{y}\rangle = \langle v(\Phi_j - \Phi_n),\mathbf{v}\rangle$ for $j = k+1,\dots,n$. Note that $\langle v(\Phi_1),\mathbf{x}\rangle = u^{(1)}$ and $\langle v(\Phi_n),\mathbf{y}\rangle = v^{(1)}$ for $\mathbf{u} = (u^{(1)},u^{(2)})$ and $\mathbf{v} = (v^{(1)},v^{(2)})$, respectively. Similarly to the proof of Lemma 3, we now introduce independent copies $(R_1,\Phi_1),\dots,(R_n,\Phi_n)$ of

(R_0, Φ_0) and independent copies $\Xi_1, \dots, \Xi_n$ of $\Xi_0 = [-R_0, R_0]$. Then the product $w_{\varrho\mathbf{x}}(p) \prod_{i=2}^{k} w_{\varrho\mathbf{x}}(p_i+p) \times \prod_{j=k+1}^{n} w_{\varrho\mathbf{y}}(p_j + p)$ can be expressed as the expectation

$$\mathbf{E}\left(\mathbf{1}_{\Xi_1+\varrho\langle\Phi_1,\mathbf{x}\rangle}(p_1) \prod_{i=2}^{k} \mathbf{1}_{\Xi_i+\varrho\langle v(\Phi_i),\,\mathbf{x}\rangle}(p_i + p_1) \prod_{j=k+1}^{n} \mathbf{1}_{\Xi_j+\varrho\langle v(\Phi_j),\,\mathbf{y}\rangle}(p_j + p_1) \right),$$

which, together with the above change of the variables $\mathbf{x}, \mathbf{y}$, allows us to express $I_{n,k}^{(\varrho)}(K)$ as follows:

$$\begin{aligned}
&\mathbf{E} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^1}\int_{\mathbb{R}^{n-1}} \varrho \Bigg(\mathbf{1}_{\Xi_1+\varrho u^{(1)}}(p) \prod_{i=2}^{k} \mathbf{1}_{\Xi_i+\varrho\langle v(\Phi_i-\Phi_1),\,\mathbf{u}\rangle}(p_i + p) \prod_{j=k+1}^{n-1} \mathbf{1}_{\Xi_j+\varrho\langle v(\Phi_j-\Phi_n),\,\mathbf{v}\rangle}(p_j + p) \\
&\qquad\qquad \times \mathbf{1}_{\Xi_n}\big(p_n + p - \varrho v^{(1)}\big) \Bigg) \big|\gamma_{\mathrm{red}}^{(n)}\big|(\mathrm{d}\mathbf{p}_{2,n})\,\mathrm{d}p \mathbf{1}_{KO(-\Phi_1)}(\mathbf{u}) \mathbf{1}_{KO(-\Phi_n)}(\mathbf{v})\,\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v} \\
&= \mathbf{E} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^1}\int_{\mathbb{R}^{n-1}} \prod_{i=2}^{k} \mathbf{1}_{\Xi_i+\langle v(\Phi_i-\Phi_1),\,(z^{(1)}+\varrho v^{(1)},\,\varrho z^{(2)})\rangle-\varrho v^{(1)}}(p_i + p) \prod_{j=k+1}^{n-1} \mathbf{1}_{\Xi_j+\varrho\langle v(\Phi_j-\Phi_n),\,\mathbf{v}\rangle-\varrho v^{(1)}}(p_j + p) \\
&\qquad\qquad \times \mathbf{1}_{\Xi_n}(p_n + p) \big|\gamma_{\mathrm{red}}^{(n)}\big|(\mathrm{d}\mathbf{p}_{2,n})\, \mathbf{1}_{\Xi_1+z^{(1)}}(p)\,\mathrm{d}p\, \mathbf{1}_{KO(-\Phi_1)}\left(\frac{z^{(1)}}{\varrho} + v^{(1)},\, z^{(2)}\right) \\
&\qquad\qquad \times \mathbf{1}_{KO(-\Phi_n)}(\mathbf{v})\,\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{v},
\end{aligned} \tag{5.19}$$

where $\mathbf{z} = (z^{(1)}, z^{(2)})$. After replacing the first two products of indicator functions in (5.19) by 1, we get the following estimate:

$$\begin{aligned}
I_{n,k}^{(\varrho)}(K) &\leqslant \mathbf{E} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^1}\int_{\mathbb{R}^{n-1}} \big(\mathbf{1}_{-\Xi_1+p}\big(z^{(1)}\big)\, \mathbf{1}_{\Xi_n-p_n}(p)\big) \big|\gamma_{\mathrm{red}}^{(n)}\big|(\mathrm{d}(\mathbf{p}_{2,n}))\,\mathrm{d}p \\
&\qquad\qquad \times \mathbf{1}_{KO(-\Phi_1)}\left(\frac{z^{(1)}}{\varrho} + v^{(1)},\, z^{(2)}\right) \mathrm{d}\mathbf{z}\, \mathbf{1}_{KO(-\Phi_n)}(\mathbf{v})\,\mathrm{d}\mathbf{v} \\
&\leqslant \mathrm{diam}(K)|K|_2 \big(\mathbf{E}|\Xi_0|_1\big)^2 \big|\gamma_{\mathrm{red}}^{(n)}\big|\big(\mathbb{R}^{n-1}\big),
\end{aligned} \tag{5.20}$$

where we have used the arguments already applied to prove (4.4). On the other hand, the product of the indicator functions in the first line of (5.19) vanishes as $\varrho \to \infty$ $\mathbf{P}$-a.s. and for almost all $\mathbf{v}, \mathbf{z}, p, \mathbf{p}_{2,n} \in \mathbb{R}^{n+4}$ with respect to the corresponding product measure. Therefore, again by Lebesgue's dominated convergence theorem,

$$\lim_{\varrho\to\infty} I_{n,k}^{(\varrho)}(K) = 0 \quad \text{for } k = 2, \dots, n,\ n \geqslant 3. \tag{5.21}$$

Next, we derive a further bound of $I_{n,k}^{(\varrho)}(K)$, which additionally depends on the mean thickness $\mathbf{E}|\Xi_0|_1 = 2\,\mathbf{E}R_0$ of the typical cylinder. For this reason, we need the Radon–Nikodym density $|c_{\mathrm{red}}^{(n)}|$ of $|\gamma_{\mathrm{red}}^{(n)}|$ with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$. Hence by using Fubini's theorem we replace integral (5.16) over $\mathbb{R}^{n-1}$ by two iterated integrals. The first integral over $\mathbf{p}_{2,\,n-1} \in \mathbb{R}^{n-2}$ can be estimated by Hölder's inequality as follows:

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n-2}} \prod_{i=2}^{k} \mathbf{1}_{\Xi_i+\langle v(\Phi_i-\Phi_1),(z^{(1)}+\varrho v^{(1)},\varrho z^{(2)})\rangle-\varrho v^{(1)}-p}(p_i) \prod_{j=k+1}^{n-1} \mathbf{1}_{\Xi_j+\varrho\langle v(\Phi_j-\Phi_n),w\rangle-\varrho v^{(1)}-p}(p_j) \big|c_{\mathrm{red}}^{(n)}(\mathbf{p}_{2,n})\big| \,\mathrm{d}\mathbf{p}_{2,n-1} \\
&\quad \leqslant \Bigg(\int\limits_{\mathbb{R}^{n-2}} \prod_{i=2}^{k} \mathbf{1}_{\Xi_i+\langle v(\Phi_i-\Phi_1),\,(z^{(1)}+\varrho v^{(1)},\,\rho z^{(2)})\rangle-\varrho v^{(1)}-p}(p_i) \\
&\qquad \times \prod_{j=k+1}^{n-1} \mathbf{1}_{\Xi_j+\varrho\langle v(\Phi_j-\Phi_n),\,\mathbf{v}\rangle-\varrho v^{(1)}-p}(p_j)\,\mathrm{d}\mathbf{p}_{2,n-1}\Bigg)^{(q-1)/q} \Bigg(\int\limits_{\mathbb{R}^{n-1}} \big|c_{\mathrm{red}}^{(n)}(\mathbf{p}_{2,n-1},p_n)\big|^q \,\mathrm{d}\mathbf{p}_{2,n-1}\Bigg)^{1/q} \\
&\quad = \Bigg(\prod_{i=2}^{n-1} |\Xi_i|_1\Bigg)^{(q-1)/q} \big\|c_{\mathrm{red}}^{(n)}(\cdot,p_n)\big\|_q
\end{aligned} \tag{5.22}$$

for all $q > 1$ and any fixed $p_n \in \mathbb{R}^1$, where $\|c_{\mathrm{red}}^{(n)}(\cdot,p_n)\|_q$ coincides with the term in front of the equality sign in (5.22). Combining estimates (5.16) and (5.22) with $|g(p,\varphi)\cap K|_1 \leqslant \mathrm{diam}(K)$ for $(p,\varphi)\in\mathbb{R}^1\times[0,\pi]$, $\int_{\mathbb{R}^1} |g(p,\varphi)\cap K|_1\,\mathrm{d}p = |K|_2$, changing the order of integration, and finally applying Lyapunov's inequality, we arrive at

$$\begin{aligned}
I_{n,k}^{(\varrho)}(K) &\leqslant \mathbf{E} \int\limits_{\mathbb{R}^1}\int\limits_{\mathbb{R}^1}\int\limits_{\mathbb{R}^1}\int\limits_{\mathbb{R}^1} \Bigg(\prod_{i=2}^{n-1}|\Xi_i|_1\Bigg)^{(q-1)/q} \big\|c_{\mathrm{red}}^{(n)}(\cdot,p_n)\big\|_q \mathbf{1}_{-\Xi_1+p}(z_1)\mathbf{1}_{\Xi_n-p_n}(p) \\
&\qquad \times \bigg|g\bigg(\frac{z_1}{\varrho}+v^{(1)},\,\Phi_1\bigg)\cap K\bigg|_1 \big|g(v^{(1)},\Phi_n)\cap K\big|_1 \,\mathrm{d}p\,\mathrm{d}p_n\,\mathrm{d}z^{(1)}\,\mathrm{d}v^{(1)} \\
&\leqslant \mathrm{diam}(K)|K|_2 \int\limits_{\mathbb{R}^1} \big\|c_{\mathrm{red}}^{(n)}(\cdot,p_n)\big\|_q\,\mathrm{d}p_n \big(\mathbf{E}|\Xi_0|_1\big)^{(n(q-1)+2)/q}.
\end{aligned}$$

The latter estimate reveals that the limit (5.21) remains true if instead of $\|\gamma_{\mathrm{red}}^{(n)}\|_{TV} < \infty$, the L_q^*-norm $\|c_{\mathrm{red}}^{(n)}\|_q^* := \int_{\mathbb{R}^1} \|c_{\mathrm{red}}^{(n)}(\cdot,p)\|_q\,\mathrm{d}p$ is finite for some $q > 1$ and $n \geqslant 3$. Furthermore, in view of estimates (5.20) and (5.22), we see that

$$T_{n,1}^{(\varrho)}(K) = \lambda \sum_{k=1}^{n-1} \binom{n}{k} I_{n,k}^{(\varrho)}(K) \leqslant \lambda\,\mathrm{diam}(K)|K|_2\big(2^n-2\big)\big(\mathbf{E}|\Xi_0|_1\big)^{n(q-1)/q+2/q}\big\|c_{\mathrm{red}}^{(n)}\big\|_q^*$$

and

$$\begin{aligned}
\sum_{n\geqslant 3} \frac{T_{n,1}^{(\varrho)}(K)}{n!} &\leqslant \lambda a_q^*\big(\mathbf{E}|\Xi_0|_1\big)^{2/q}\,\mathrm{diam}(K)|K|_2 \sum_{n\geqslant 3}\big(2b_q^*\big(\mathbf{E}|\Xi_0|_1\big)^{(q-1)/q}\big)^n \\
&\leqslant \frac{\lambda a_q^*(\mathbf{E}|\Xi_0|_1)^{2/q}\,\mathrm{diam}(K)|K|_2}{1-2\,b_q^*\,(\mathbf{E}|\Xi_0|_1)^{1-1/q}},
\end{aligned}$$

provided that the strong L_q^*-Brillinger mixing condition with $b_q^*(\mathbf{E}|\Xi_0|_1)^{1-1/q} < 1/2$ is satisfied.

Next, we derive two different bounds for the sum $T_{n,2}^{(\varrho)}(K)$ defined in (5.17). For doing this, in analogy to $I_{n,k}^{(\varrho)}(K)$, we need uniform bounds (only depending on n) of

$$J_{n,k}^{(\varrho)}(p) := \int\limits_{\mathbb{R}^{n-1}} \prod_{i=2}^{k} w_{\varrho\mathbf{x}}(p_i+p) \prod_{j=k+1}^{n} w_{\varrho\mathbf{y}}(p_j+p) \left|\gamma_{\mathrm{red}}^{(n)}\right|(\mathrm{d}\mathbf{p}_{2,n}) \quad \text{for } 2 \leqslant k \leqslant n.$$

Since here the integrand of the integral vanishes as $\varrho \to \infty$, it follows, again by Lebesgue's dominated convergence theorem, that

$$\lim_{\varrho\to\infty} J_{n,k}^{(\varrho)}(p) = 0 \quad \text{and} \quad J_{n,k}^{(\varrho)}(p) \leqslant \left|\gamma_{\mathrm{red}}^{(n)}\right|\left(\mathbb{R}^{n-1}\right) \quad \text{for } k = 2,\dots,n,\ n \geqslant 3. \tag{5.23}$$

Furthermore, if $\|c_{\mathrm{red}}^{(n)}\|_q < \infty$ for some $q > 1$, then we obtain the alternative estimate

$$\begin{aligned} J_{n,k}^{(\varrho)}(p) &= \mathbf{E} \int\limits_{\mathbb{R}^{n-1}} \prod_{i=2}^{k} \mathbf{1}_{\Xi_i+\varrho\langle v(\Phi_i),\mathbf{x}\rangle-p}(p_i) \prod_{j=k+1}^{n} \mathbf{1}_{\Xi_j+\varrho\langle v(\Phi_j),\mathbf{y}\rangle-p}(p_j)\, c_{\mathrm{red}}^{(n)}(\mathbf{p}_{2,n})\,\mathrm{d}(\mathbf{p}_{2,n}) \\ &\leqslant \mathbf{E}\prod_{i=2}^{n} |\Xi_i|^{(q-1)/q} \Big(\int\limits_{\mathbb{R}^{n-1}} \left|c_{\mathrm{red}}^{(n)}(\mathbf{p}_{2,n})\right|^q \mathrm{d}\mathbf{p}_{2,n} \Big)^{1/q} \leqslant \left(\mathbf{E}|\Xi_0|_1\right)^{(n-1)(q-1)/q} \left\|c_{\mathrm{red}}^{(n)}\right\|_q. \end{aligned} \tag{5.24}$$

Further, from the definition of $T_{n,2}^{(\varrho)}(K)$ in (5.17) and the estimate $J_\varrho(K) \leqslant 2\,\mathrm{diam}(K)\,|K|_2\,\mathbf{E}|\Xi_0|_1^2$ obtained in the proof of Lemma 6, we arrive at

$$\begin{aligned} T_{n,2}^{(\varrho)}(K) &\leqslant \lambda n \sum_{k=1}^{n} \binom{n-1}{k-1} \int\limits_{K^2}\int\limits_{\mathbb{R}^1} \varrho w_{\varrho\mathbf{x},\varrho\mathbf{y}}^{\cap}(p_1)\,\mathrm{d}p_1\,\mathrm{d}(\mathbf{x},\mathbf{y}) \max_{2\leqslant k\leqslant n} \sup_{p\in\mathbb{R}^1} J_{n,k}^{(\varrho)}(p) \\ &= \lambda n 2^{n-1} J_\varrho(K) \max_{2\leqslant k\leqslant n} \sup_{p\in\mathbb{R}^1} J_{n,k}^{(\varrho)}(p) \\ &\leqslant \lambda n 2^n\,\mathrm{diam}(K)|K|_2\mathbf{E}|\Xi_0|_1^2 \left(\mathbf{E}|\Xi_0|\right)^{(n-1)(q-1)/q} \left\|c_{\mathrm{red}}^{(n)}\right\|_q. \end{aligned}$$

Under the assumption that $\Psi \sim P$ is strongly Brillinger mixing with $b < 1/2$ or strongly L_q-Brillinger mixing with $b_q(\mathbf{E}|\Xi_0|_1)^{1-1/q} < 1/2$, we obtain the estimates

$$\sum_{n\geqslant 3} \frac{T_{n,2}^{(\varrho)}(K)}{n!} \leqslant 2\lambda ab\mathbf{E}|\Xi_0|_1^2\,\mathrm{diam}(K)|K|_2 \sum_{n\geqslant 3} n\left(2b\right)^{n-1} \leqslant \frac{2\lambda ab\mathbf{E}|\Xi_0|_1^2\,\mathrm{diam}(K)|K|_2}{(1-2b)^2}$$

and

$$\sum_{n\geqslant 3} \frac{T_{n,2}^{(\varrho)}(K)}{n!} \leqslant \frac{2\lambda a_q b_q\mathbf{E}|\Xi_0|_1^2\,\mathrm{diam}(K)|K|_2}{(1-2b_q(\mathbf{E}|\Xi_0|_1)^{1-1/q})^2}, \quad \text{respectively.}$$

Finally, by summarizing the above-proved relations (5.20)–(5.24) and the convergence of the series $\sum_{n\geqslant 3} T_{n,i}^{(\varrho)}(K)/n!$ for $i = 1, 2$ we get (5.15). It remains to show that the second limit in (5.8) vanishes.

Putting $\widetilde{L}(\mathbf{x},\mathbf{y}) := L(\mathbf{x},\mathbf{y}) + T_1(\mathbf{x},\mathbf{y}) - T_2(\mathbf{x},\mathbf{y})/2$, we get that

$$\varrho \int_{K^2} \bigl(L(\varrho\mathbf{x},\varrho\mathbf{y})\bigr)^2 \,\mathrm{d}(\mathbf{x},\mathbf{y}) \leqslant 2\int_{K^2} \varrho\bigl(\widetilde{L}(\varrho\mathbf{x},\varrho\mathbf{y})\bigr)^2 \,\mathrm{d}(\mathbf{x},\mathbf{y}) + 4\int_{K^2} \varrho\bigl(T_1(\varrho\mathbf{x},\varrho\mathbf{y})\bigr)^2 \,\mathrm{d}(\mathbf{x},\mathbf{y}) + \int_{K^2} \varrho\bigl(T_2(\varrho\mathbf{x},\varrho\mathbf{y})\bigr)^2 \,\mathrm{d}(\mathbf{x},\mathbf{y}). \tag{5.25}$$

From (5.18) we easily see that $|\widetilde{L}(\varrho\mathbf{x},\varrho\mathbf{y})|$ and $|L(\varrho\mathbf{x},\varrho\mathbf{y})|$ have the same bound. Hence by combining (5.15) and (5.18) we see that the first integral on the right-hand side of (5.25) converges to zero as $\varrho \to \infty$. We still have to show that $\int_{K^2} \varrho(T_i(\varrho\mathbf{x},\varrho\mathbf{y}))^2\,\mathrm{d}(\mathbf{x},\mathbf{y}) \to 0$ as $\varrho \to \infty$ for $i = 1,2$. We rewrite (5.12) for $n = 1$ by introducing independent pairs $(\Xi_0,\Phi_0), (\Xi_1,\Phi_1)$ and polar coordinates $\mathbf{y} = r\,v(\psi)$. As in the proof of Lemma 6, we substitute $\psi = \arccos(z/r)$ for $z \in [-r,r]$ and use the function $h(r,z,\varphi)$ with $v(\psi \pm \pi) = \mp v(\psi)$, which leads to

$$\begin{aligned}
&\int_{K^2} \varrho\bigl(T_1(\varrho\mathbf{x},\varrho\mathbf{y})\bigr)^2 \,\mathrm{d}(\mathbf{x},\mathbf{y}) \\
&\quad= \lambda^2 \int_{K^2}\int_{\mathbb{R}^2} \varrho w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_1) w^{\cap}_{\varrho\mathbf{x},\varrho\mathbf{y}}(p_2)\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}(\mathbf{x},\mathbf{y}) \\
&\quad= \lambda^2 \mathbf{E}\int_{\mathbb{R}^2} \varrho \mathbf{1}_{K\oplus(-K)}(\mathbf{y})\bigl|K\cap(K-\mathbf{y})\bigr|_2 \prod_{i=0}^{1}\bigl|\Xi_i \cap \bigl(\Xi_i + \varrho\langle v(\Phi_i),\, \mathbf{y}\rangle\bigr)\bigr|_1\,\mathrm{d}\mathbf{y} \\
&\quad= \lambda^2 \mathbf{E}\int_0^{2\pi}\int_0^{\infty} \varrho\bigl|K\cap\bigl(K - r\,v(\psi)\bigr)\bigr|_2 \prod_{i=0}^{1}\bigl|\Xi_i \cap \bigl(\Xi_i + \varrho r\cos(\psi-\Phi_i)\bigr)\bigr|_1 r\,\mathrm{d}r\,\mathrm{d}\psi \\
&\quad= 2\lambda^2 \mathbf{E}\int_0^{\pi}\int_0^{\infty} \bigl|K\cap\bigl(K - r\,v(\psi+\Phi_0)\bigr)\bigr|_2 \bigl|\Xi_0\cap(\Xi_0 + \varrho r\cos\psi)\bigr|_1 \\
&\qquad\qquad \times \bigl|\Xi_1\cap\bigl(\Xi_1 + \varrho r\cos(\psi+\Phi_0-\Phi_1)\bigr)\bigr|_1 \varrho r\,\mathrm{d}r\,\mathrm{d}\psi \\
&\quad\left(\text{with } r\cos\left(\arccos\frac{z}{r} + \Phi_0 - \Phi_1\right) = z\cos(\Phi_0-\Phi_1) - \sqrt{r^2-z^2}\sin(\Phi_0-\Phi_1)\right) \\
&\quad= \lambda^2 \mathbf{E}\int_0^{\infty}\int_{-r}^{r} \bigl|K\cap\bigl(K - h(r,z,\Phi_0)\bigr|_2 \bigl|\Xi_1\cap\bigl(\Xi_1 + \varrho z\cos(\Phi_0-\Phi_1) - \varrho\sqrt{r^2-z^2}\sin(\Phi_0-\Phi_1)\bigr)\bigr|_1 \\
&\qquad\qquad \times \bigl|\Xi_0\cap(\Xi_0+\varrho z)\bigr|_1 \frac{\varrho\,\mathrm{d}z\,\mathrm{d}(r^2)}{\sqrt{r^2-z^2}} \quad \bigl(\text{with new variables } s = r^2 - z^2 \text{ and } u = \varrho z\bigr) \\
&\quad\leqslant \lambda^2 |K|_2 \mathbf{E}\int_{\mathbb{R}^1}\int_0^{r_K} \bigl|\Xi_1\cap\bigl(\Xi_1 + u\cos(\Phi_0-\Phi_1) - \varrho\sqrt{s}\sin(\Phi_0-\Phi_1)\bigr)\bigr|_1 \bigl|\Xi_0\cap(\Xi_0+u)\bigr|_1 \frac{\mathrm{d}s}{\sqrt{s}}\,\mathrm{d}u.
\end{aligned} \tag{5.26}$$

Here the integrals could be interchanged taking into account that $\|h(r,z,\varphi)\| = r \leqslant r_K$ as in (4.5). Since the integrand of the double integral in (5.26) is bounded by $|\Xi_1|_1|\Xi_0\cap(\Xi_0+u)|_1/\sqrt{s}$ and vanishes as $\varrho \to \infty$ ($\mathbf{P}$-a.s.), by Lebesgue's dominated convergence theorem and $\mathbf{E}|\Xi_0|_1^2 < \infty$ it follows that the integrals in line (4.5) vanishes as $\varrho \to \infty$. In view of (5.13) and (5.14), the proof of Theorem 2 is complete if we show that

$$\widehat{T_2}^{(\varrho)}(K) := \int\limits_{K^2} \varrho\Big(\int\limits_{\mathbb{R}^2} w_{\varrho\mathbf{x}}(p_1) w_{\varrho\mathbf{y}}(p_2)\, \gamma^{(2)}(\mathrm{d}\mathbf{p}_{1,2}) \Big)^2 \mathrm{d}(\mathbf{x},\mathbf{y}) \underset{\varrho\to\infty}{\longrightarrow} 0. \tag{5.27}$$

As in the proof of Lemma 5, we use the realizations (ξ_i, φ_i) of the independent pairs (Ξ_i, Φ_i) for $i = 1, \dots, 4$ and omit the expectation. Again, substituting $\mathbf{x} = \mathbf{s}O(-\varphi_1)$ and $\mathbf{y} = \mathbf{t}O(-\varphi_2)$ with $\mathbf{s} = (s^{(1)}, s^{(2)})$ and $\mathbf{t} = (t^{(1)}, t^{(2)})$, we get $\langle v(\varphi_i), \mathbf{s}O(-\varphi_1)\rangle = \langle v(\varphi_i - \varphi_1), \mathbf{s}\rangle$ for $i = 1, 3$ and $\langle v(\varphi_i), \mathbf{t}O(-\varphi_2)\rangle = \langle v(\varphi_i - \varphi_2), \mathbf{t}\rangle$ for $i = 2, 4$. Applying Fubini's theorem together with $\int_{\mathbb{R}^1} \mathbf{1}_{KO(\varphi)}(\mathbf{t})\,\mathrm{d}t^{(2)} = |g(t^{(1)}, \varphi) \cap K|_1 \leqslant \mathrm{diam}(K)$ and $\int_{\mathbb{R}^1} |g(t^{(1)}, \varphi) \cap K|_1\,\mathrm{d}t^{(1)} = |K|_2$, we arrive at

$$\begin{aligned}
\widehat{T_2}^{(\varrho)}(K) &= \lambda^2 \varrho \int\limits_{\mathbb{R}^4}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \mathbf{1}_{\xi_1+\varrho(s^{(1)}-t^{(1)})}(p_1) \mathbf{1}_{\xi_2}(p_2+p_1) \mathbf{1}_{\xi_3+\varrho\langle v(\varphi_3-\varphi_1),\,\mathbf{s}\rangle}(p_3) \mathbf{1}_{\xi_4+\varrho\langle v(\varphi_4-\varphi_2),\,\mathbf{t}\rangle}(p_4+p_3) \\
&\qquad \times \mathbf{1}_{KO(\varphi_1)}(\mathbf{s}) \mathbf{1}_{KO(\varphi_2)}(\mathbf{t})\,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{t}\, \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_4)\,\mathrm{d}p_3\, \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_2)\,\mathrm{d}p_1 \\
&= \lambda^2 \int\limits_{\mathbb{R}^4}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \mathbf{1}_{\xi_1+s^{(1)}}(p_1) \mathbf{1}_{\xi_2}(p_2+p_1) \mathbf{1}_{\xi_3+\langle v(\varphi_3-\varphi_1),\,(s^{(1)}+\varrho t^{(1)},\,\varrho\, s^{(2)})\rangle}(p_3) \\
&\qquad \times \gamma^{(2)}_{\mathrm{red}}\big(\xi_4 + \varrho\big\langle v(\varphi_4-\varphi_2),\,\mathbf{t}\big\rangle - p_3\big) \mathbf{1}_{KO(\varphi_1)}\Big(\frac{s^{(1)}}{\varrho} + t^{(1)},\, s^{(2)}\Big) \\
&\qquad \mathbf{1}_{KO(\varphi_2)}\big(t^{(1)}, t^{(2)}\big)\,\mathrm{d}\big(s^{(1)}, s^{(2)}\big)\,\mathrm{d}\big(t^{(1)}, t^{(2)}\big)\,\mathrm{d}p_3\, \gamma^{(2)}_{\mathrm{red}}(\mathrm{d}p_2)\,\mathrm{d}p_1 \\
&\leqslant \lambda^2 \int\limits_{\mathbb{R}^4}\int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^1} \mathbf{1}_{\xi_3+\langle v(\varphi_3-\varphi_1),\,(s^{(1)}+\varrho t^{(1)},\,\varrho s^{(2)})\rangle}(p_3) \big|\gamma^{(2)}_{\mathrm{red}}\big|\big(\xi_4 + \varrho\big\langle v(\varphi_4-\varphi_2),\,\mathbf{t}\big\rangle - p_3\big)\,\mathrm{d}p_3 \\
&\qquad \times \mathbf{1}_{\xi_2}(p_2+p_1) \mathbf{1}_{-\xi_1+\varrho t^{(1)}+p_1}\big(s^{(1)}\big)\, \big|\gamma^{(2)}_{\mathrm{red}}\big|(\mathrm{d}p_2)\,\mathrm{d}p_1\, \mathbf{1}_{KO(\varphi_1)}\Big(\frac{s^{(1)}}{\varrho}, s^{(2)}\Big) \\
&\qquad \times \mathbf{1}_{KO(\varphi_2)}(\mathbf{t})\,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{t}.
\end{aligned} \tag{5.28}$$

As $\varrho \to \infty$, the inner integral over p_3 in (5.28) vanishes and is bounded by $|\xi_3|_1\, |\gamma^{(2)}_{\mathrm{red}}|(\mathbb{R}^1)$. The outer integrals over $\mathbf{s}, \mathbf{t}, p_1, p_2$ are bounded by $|\xi_1|_1\, |\xi_2|_1\, |\gamma^{(2)}_{\mathrm{red}}|(\mathbb{R}^1)\, \mathrm{diam}(K)\, |K|_2$. Hence by Lebesgue's dominated convergence theorem the limit (5.27) is shown. Together with (5.15) and (5.18), the second limit in (5.8) is equal to zero. Thus the proof of Theorem 2 is complete. □

Remark 3. Note that in Theorem 1 (resp., Theorem 2) the interval $\Xi_0 := [-R_0, R_0]$ with $\mathbf{E}R_0^k < \infty$ can be replaced by a finite union of random closed intervals $\Xi_0 \subset \mathbb{R}^1$ satisfying $\mathbf{E}|\Xi_0|_1^k < \infty$ for $k = 1$ (resp., $k = 2$). This extension is based on the definition of a process of cylinders with nonconvex bases; see, for example, [20]. Furthermore, the exponential shape of the PGF of a Poisson cluster PP $\Psi \sim P$ (see [2,3]) allows us to simplify the function $L(\mathbf{x}, \mathbf{y})$ in (5.6) and to reduce the Brillinger-mixing-type conditions in Theorems 1 and 2.

6 Conclusion

Strong Brillinger mixing with $b < 1/2$ is a rather restrictive condition for the one-dimensional PP $\Psi \sim P$. Equivalently formulated, the power series $\sum_{n=2}^{\infty} (z^n/n!)|\gamma^{(n)}_{\mathrm{red}}|(\mathbb{R}^{n-1})$ is analytic in the interior of the disk $b(\mathbf{o}, 2)$ in the complex plane. Such a strong condition has been used for statistical analysis of point processes in [4]. The Gauss–Poisson process, Poisson cluster processes with a finite number of nonvanishing cumulant measures, and, among them, certain Neyman–Scott processes (see, e.g., [2]) satisfy this condition. On the other hand, if a PP $\Psi \sim P$ is strongly L_q-Brillinger mixing (resp., strongly L_q^*-Brillinger-mixing) for some $q > 1$ with $b_q > 0$ (resp., $b_q^* > 0$), then we can choose $\mathbf{E}|\Xi_0|_1$ sufficiently small to fulfill the assumptions of Theorem 2, which greatly expands its applicability.

Another question concerns the asymptotic normality of the scaled and centered total area $Z^{(\varrho)}(K) := \varrho^{-3/2}(|\Xi \cap \varrho K|_2 - \mathbf{E}|\Xi \cap \varrho K|_2)$ of the union set (1.1) in ϱK as $\varrho \to \infty$. To achieve this goal, we need to find conditions (as mild as possible, but certainly stronger than in Theorem 2) implying that all cumulants $\mathrm{Cum}_k(Z^{(\varrho)}(K)) = \varrho^{-3k/2}\,\mathrm{Cum}_k(|\Xi \cap \varrho K|_2)$ of order $k \geqslant 3$ vanish as $\varrho \to \infty$. With the notation and by using and extending some results in Section 3 (in particular, Lemma 2) we easily see that $\mathrm{Cum}_k(Z^{(\varrho)}(K)) \to 0$ as $\varrho \to \infty$ for any fixed $k \geqslant 3$ if and only if

$$\varrho^{k/2} \sum_{\ell=1}^{k} \frac{(-1)^{\ell-1}}{\ell} \sum_{\substack{k_1+\cdots+k_\ell=k \\ k_i \geqslant 1,\ i=1\ldots,\ell}} \frac{k!}{k_1!\cdots k_\ell!} \prod_{j=1}^{\ell} \int_{K^{k_j}} G_P\big[1 - w^{\cup}_{\varrho x_1,\ldots,\varrho x_{k_j}}(\cdot)\big]\,\mathrm{d}(x_1,\ldots,x_{k_j}) \underset{\varrho\to\infty}{\longrightarrow} 0. \qquad (6.1)$$

A profound modification of the recursive technique applied in Section 2 of [13] to prove (6.1) for Poisson CPs seems to be promising. The details will be subject of a separate paper.

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