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On ordered Ramsey numbers of matchings versus triangles

(Extended abstract)

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Abstract

For graphs $G^<$ and $H^<$ with linearly ordered vertex sets, the *ordered Ramsey number* $r_<(G^<, H^<)$ is the smallest $N \in \mathbb{N}$ such that any red-blue coloring of the edges of the complete ordered graph $K_N^<$ on N vertices contains either a blue copy of $G^<$ or a red copy of $H^<$. Motivated by a problem of Conlon, Fox, Lee, and Sudakov (2017), we study the numbers $r_<(M^<, K_3^<)$ where $M^<$ is an n-vertex ordered matching.

We prove that almost all n-vertex ordered matchings $M^<$ with interval chromatic number 2 satisfy $r_<(M^<, K_3^<) \in \Omega((n/\log n)^{5/4})$ and $r_<(M^<, K_3^<) \in O(n^{7/4})$, improving a recent result by Rohatgi (2019). We also show that there are n-vertex ordered matchings $M^<$ with interval chromatic number at least 3 satisfying $r_<(M^<, K_3^<) \in \Omega((n/\log n)^{4/3})$, which asymptotically matches the best known lower bound on these ordered Ramsey numbers for general n-vertex ordered matchings.

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1 Introduction

For graphs G and H, their *Ramsey number* $r(G, H)$ is the smallest positive integer N such that any coloring of the edges of K_N with colors red and blue contains either G as a

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subgraph with all edges colored red or H as a subgraph with all edges colored blue. The existence of these numbers was proved by Ramsey [18] and later independently by Erdős and Szekeres [13]. The study of the growth rate of these numbers with respect to the number of vertices of G and H is a classical part of combinatorics and plays a central role in Ramsey theory.

Motivated by several applications from discrete geometry and extremal combinatorics, various researchers [3, 4, 6, 10, 16, 19] started investigating an ordered variant of Ramsey numbers. An *ordered graph* $G^<$ is a graph G together with a linear ordering of its vertex set. An ordered graph $G^<$ is an *ordered subgraph* of another ordered graph $H^<$ if G is a subgraph of H and the vertex ordering of G is a suborder of the vertex ordering of H. The *ordered Ramsey number* $r_<(G^<, H^<)$ of ordered graphs $G^<$ and $H^<$ is the smallest positive integer N such that any coloring of the edges of the complete ordered graph $K_N^<$ on N vertices with colors red and blue contains either $G^<$ as an ordered subgraph with all edges red or $H^<$ as an ordered subgraph with all edges blue.

It is known that the ordered Ramsey numbers always exist and that they can behave very differently from the unordered Ramsey numbers. For example, there are ordered *matchings* $M^<$ (that is, 1-regular ordered graphs) on n vertices for which $r_<(M^<, M^<)$ grows superpolynomially in n, in particular, we have $r_<(M^<, M^<) \in 2^{\Omega(\log^2 n/\log\log n)}$ [3, 10] while $r(G,G)$ is linear for all graphs G with bounded maximum degrees [7]. The superpolynomial bound obtained for ordered matchings is almost tight for sparse graphs as, for every fixed $d \in \mathbb{N}$, every d-degenerate ordered graph $G^<$ on n vertices satisfies $r_<(G^<, G^<) \in 2^{O(\log^2 n)}$ [10].

One of the most interesting cases for ordered Ramsey numbers is the study of the growth rate of $r_<(M^<, K_3^<)$ where $M^<$ is an ordered matching on n vertices as this is one of the first non-trivial cases where the exact asymptotics is not known. Conlon, Fox, Lee, and Sudakov [10] observed that the classical bound $r(K_n, K_3) \in O(n^2/\log n)$ immediately gives $r_<(M^<, K_3^<) \in O(n^2/\log n)$. On the other hand, they showed that there exists a positive constant c such that, for all even positive integers n, there is an ordered matching $M^<$ on n vertices with

$$r_<(M^<, K_3^<) \geq c\left(\frac{n}{\log n}\right)^{4/3}. \tag{1}$$

Conlon, Fox, Lee, and Sudakov expect that the upper bound $r_<(M^<, K_3^<) \leq O(n^2/\log n)$ is far from optimal and posed the following open problem [10], which is also mentioned in a survey on recent developments in graph Ramsey theory [11].

Problem 1 ([10])**.** *Does there exist an $\epsilon > 0$ such that for any ordered matching $M^<$ on n vertices $r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$?*

Problem 1 is one of the most important questions in the theory of ordered Ramsey numbers as in order to get a subquadratic upper bound on $r_<(M^<, K_3^<)$ one has to be able to employ the sparsity of $M^<$ since the bound $r(K_n, K_3) \in O(n^2/\log n)$ is asymptotically tight by a famous result of Kim [15]. Being able to use the sparsity of $M^<$ and thus distinguish $M^<$ from $K_n^<$ could help in numerous problems on ordered Ramsey numbers. However, this is difficult as some ordered matchings $M^<$ can be used to approximate the behavior of complete graphs, which is the reason why the numbers $r_<(M^<, M^<)$ can grow superpolynomially.

Some partial progress on Problem 1 was recently made by Rohatgi [19] who considered ordered matchings with bounded interval chromatic number. The *interval chromatic number* $\chi_<(G^<)$ of an ordered graph $G^<$ is the minimum number of intervals the vertex set of $G^<$ can be partitioned into so that there is no edge of $G^<$ with both vertices in the same interval. The interval chromatic number can be understood as an analogue of the chromatic number for ordered graphs as, for example, there is a variant of the Erdős–Stone–Simonovits theorem for ordered graphs [17] that is expressed in terms of the interval chromatic number.

Rohatgi [19] showed that the subquadratic bound on $r_<(M^<, K_3^<)$ holds for almost all ordered matchings with interval chromatic number 2 by proving the following result.

Theorem 1 ([19]). *There is a constant c such that for every even positive integer n, if an ordered matching $M^<$ on n vertices with $\chi_<(M^<) = 2$ is picked uniformly at random, then with high probability*

$$r_<(M^<, K_3^<) \leq cn^{24/13}.$$

Motivated by Problem 1, we study the numbers $r_<(M^<, K_3^<)$ for ordered matchings with bounded interval chromatic number. We strengthen some bounds by Rohatgi [19] and by Conlon, Fox, Lee, and Sudakov [10], obtaining a new partial progress on Problem 1.

From now on, we omit floor and ceiling signs whenever they are not essential. For $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1, \dots, n\}$. All logarithms in this paper are base 2.

2 Our results

We try to tackle the first non-trivial instance of Problem 1 by considering the typical behavior of the numbers $r_<(M^<, K_3^<)$ for ordered matchings with interval chromatic number 2. As far as we know, there is no non-trivial lower bound in this case. In his paper, Rohatgi [19] mentions the problem of obtaining lower bounds that would come closer to the upper bound from Theorem 1. As our first result, we prove the first superlinear lower bound for this case.

Theorem 2. *There exists a positive constant c such that, for all even positive integers n, if an ordered matching $M^<$ on n vertices with $\chi_<(M^<) = 2$ is picked uniformly at random, then with high probability*

$$r_<(M^<, K_3^<) \geq c\left(\frac{n}{\log n}\right)^{5/4}.$$

We also show that this lower bound can be improved for ordered matchings $M^<$ with $\chi_<(M^<) > 2$.

Theorem 3. *For every integer $k \geq 3$, there exists a positive constant $c = c(k)$ such that, for all even positive integers n, there exists an ordered matching $M^<$ on n vertices with $\chi_<(M^<) = k$ satisfying*

$$r_<(M^<, K_3^<) \geq c\left(\frac{n}{\log n}\right)^{4/3}.$$

Note that the lower bound from Theorem 3 asymptotically matches the bound (1) by Conlon, Fox, Lee, and Sudakov [10]. Thus, the best known lower bound on $r_<(M^<, K_3^<)$ for general ordered matchings $M^<$ can be obtained also for ordered matchings with bounded interval chromatic number as long as this number is at least 3. The proofs of Theorems 2 and 3 are probabilistic and are based on ideas used by Conlon, Fox, Lee, and Sudakov [10].

Rohatgi [19] was also interested in determining how far from the truth the exponent 24/13 from Theorem 1 is. We narrow the gap there by providing the following upper bound that strengthens Theorem 1.

Theorem 4. *There is a constant c such that for every even positive integer n, if an ordered matching $M^<$ on n vertices with $\chi_<(M^<) = 2$ is picked uniformly at random, then with high probability*

$$r_<(M^<, K_3^<) \leq cn^{7/4}.$$

Note that the difference between the exponent in the lower bound from Theorem 2 and the exponent in the upper bound from Theorem 4 is exactly 1/2. The sketch of the proof of Theorem 4 is in Section 4. All proofs can be found in the full version of this paper [5].

3 Open problems

Problem 1 still remains wide open, but there are many interesting intermediate questions that one could try to tackle. The following interesting conjecture was posed by Rohatgi [19].

Conjecture 1 ([19])**.** *For every integer $k \geq 2$, there is a constant $\varepsilon = \varepsilon(k) > 0$ such that*

$$r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$$

for almost every ordered matching $M^<$ on n vertices with $\chi_<(M^<) = k$.

It follows from Theorem 4 that $\varepsilon(2) \geq 1/4$. The conjecture is open for all cases with $k \geq 3$. Our results suggest that $\varepsilon(2) > \varepsilon(3)$ might hold.

Concerning the ordered matchings $M^<$ with interval chromatic number 2, even in this case the growth rate of $r_<(M^<, K_3^<)$ is not understood, so we pose the following weaker version of Problem 1.

Conjecture 2. *There exists an $\epsilon > 0$ such that for any ordered matching $M^<$ on n vertices with $\chi_<(M^<) = 2$ we have $r_<(M^<, K_3^<) \in O(n^{2-\varepsilon})$.*

In this paper, we considered the variant of this problem for random ordered matchings with interval chromatic number 2, but there is still a gap between our bounds. It would be very interesting to close it.

Problem 2. *What is the growth rate of $r_<(M^<, K_3^<)$ for uniform random ordered matchings $M^<$ on n vertices with $\chi_<(M^<) = 2$?*

It follows from our results that the answer to Problem 2 lies somewhere between $\Omega((n/\log n)^{5/4})$ and $O(n^{7/4})$. We do not know which of these bounds is closer to the truth.

4 Sketch of the proof of Theorem 4

To prove Theorem 4, we use a multi-thread scanning procedure whose variants were recently used by Cibulka and Kynčl [9], He and Kwan [14], and Rohatgi [19].

First, we associate an ordered matching $M^<$ on $[2n]$ with the permutation $\pi_{M^<}$ on $[n]$ that maps i to $j-n$ for every edge $\{i,j\}$ of $M^<$. Let χ be a red-blue coloring of the edges of $K^<_{2N}$ for some $N \in \mathbb{N}$. Let A be an $N \times N$ matrix where an entry on position $(i,j) \in [N] \times [N]$ is the color of the edge $\{i, N+j\}$ in χ.

We now describe a procedure that we use to find a red copy of $M^<$ in χ. It suffices to find an $n \times n$ submatrix of A with red entries on positions $(i, \phi_{M^<}(i))$ for $i = 1, \ldots, n$. Let $T \in \mathbb{N}$. Consider the rows $t+1, \ldots, t+n$ of A for every $t \in \{0, 1, \ldots, T-1\}$. First, we scan through the row $\pi_{M^<}(1)+t$ of A from left to right until we find a red entry in some position $(\pi_{M^<}(1)+t, j_1)$. For every $i \in \{2, \ldots, n\}$, after we have finished scanning through rows $\pi_{M^<}(1)+t, \ldots, \pi_{M^<}(i-1)+t$, we scan through the row $\pi_{M^<}(i)+t$ of A, starting from column $j_{i-1}+1$, until we find a red entry in some position $(\pi_{M^<}(i)+t, j_i)$.

We call this *multi-thread scanning* for $M^<$ and we call the set $Th(t)$ of entries of A that are revealed in step t a *thread*. Note that $Th(t)$ finds a red copy of $M^<$ if and only if some red copy of $M^<$ lies in the rows $t+1, \ldots, t+n$ of A.

For a permutation π on $[n]$, we say that a subset $C \subseteq [n]$ with $|C| = k$ is a *shift* of another subset $D \subseteq [n]$ in π if there is a positive integer Δ such that $\pi(c_i) = \pi(d_i) + \Delta$ for each $i \in [k]$ where $c_1 < \cdots < c_k$ and $d_1 < \cdots < d_k$ are the elements of C and D, respectively. Let $L(\pi)$ be the largest positive integer k for which there are sets $C, D \subseteq [n]$, each of size k, such that C is a shift of D.

The multi-thread scanning procedure yields the following result, which gives asymptotically stronger bounds than a similar result obtained by Rohatgi [19].

Theorem 5. *For $n \in \mathbb{N}$, let $M^<$ be an ordered matching on $2n$ vertices with $\chi_<(M^<) = 2$ and $L(\pi_{M^<}) \leq \ell$. If $N \geq 4n(\sqrt{n\ell}+1)$, then every red-blue coloring of the edges of $K^<_{2N}$ on $[2N]$ satisfies at least one of the following three claims:*

(a) χ contains a blue copy of $K^<_3$,

(b) χ contains a red copy of $K^<_{2n}$, or

(c) χ contains a red copy of $M^<$ between $[N]$ and $\{N+1, \ldots, 2N\}$.

For every $\varepsilon > 0$, Theorem 5 immediately implies that $r_<(M^<, K^<_3) \in O(n^{2-\varepsilon})$ for every ordered matching with $\chi_<(M^<) = 2$ and $L(\pi_{M^<}) \leq n^{1-2\varepsilon}$. It suffices to show that this is the case for uniform random ordered matchings with interval chromatic number 2. We do so by using the following result of He and Kwan [14].

Lemma 6 ([14]). *A uniform random permutation π on $[n]$ satisfies $L(\pi) \leq 3\sqrt{n}$ with high probability.*

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