Random Embeddings of Graphs: The Expected Number of Faces in Most Graphs is Logarithmic*

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Abstract

A random 2-cell embedding of a connected graph G in some orientable surface is obtained by choosing a random local rotation around each vertex. Under this setup, the number of faces or the genus of the corresponding 2-cell embedding becomes a random variable. Random embeddings of two particular graph classes – those of a bouquet of n loops and those of n parallel edges connecting two vertices – have been extensively studied and are well-understood. However, little is known about more general graphs despite their important connections with central problems in mainstream mathematics and in theoretical physics (see [Lando & Zvonkin, Graphs on surfaces and their applications, Springer 2004]). There are also tight connections with problems in computing (random generation, approximation algorithms). The results of this paper, in particular, explain why Monte Carlo methods (see, e.g., [Gross & Tucker, Local maxima in graded graphs of imbeddings, Ann. NY Acad. Sci 1979] and [Gross & Rieper, Local extrema in genus stratified graphs, JGT 1991]) cannot work for approximating the minimum genus of graphs.

In his breakthrough work ([Stahl, Permutation-partition pairs, JCTB 1991] and a series of other papers), Stahl developed the foundation of "random topological graph theory". Most of his results have been unsurpassed until today. In our work, we analyze the expected number of faces of random embeddings (equivalently, the average genus) of a graph G. It was very recently shown [Campion Loth & Mohar, Expected number of faces in a random embedding of any graph is at most linear, CPC 2023] that for any graph G, the expected number of faces is at most linear. We show that the actual expected number of faces $F(G)$ is almost always much smaller. In particular, we prove the following results:

(1) $\frac{1}{2}\ln n - 2 < \mathbb{E}[F(K_n)] \leq 3.65\ln n + o(1)$. This substantially improves Stahl's $n + \ln n$ upper bound for this case.

(2) For random graphs $G(n,p)$ ($p = p(n)$), we have $\mathbb{E}[F(G(n,p))] \leq \ln^2 n + \frac{1}{p}$.

(3) For random models $B(n,\Delta)$ containing only graphs, whose maximum degree is at most Δ, we obtain stronger bounds by showing that the expected number of faces is $\Theta(\ln n)$.

1 Introduction

1.1 Random embeddings of graphs in surfaces Every 2-cell embedding of a graph G in an (orientable) surface can be described combinatorially up to homeomorphic equivalence by using a *rotation system*. This is a set of cyclic permutations $\{R_v \mid v \in V(G)\}$, where R_v describes the clockwise cyclic order of edges incident with v in an embedding of G in an oriented surface. We refer to [33] for further details. In this way, a connected graph G, whose vertices have degrees $d(v)$ ($v \in V(G)$), admits precisely $\prod_{v\in V(G)}(d(v)-1)!$ nonequivalent 2-cell embeddings.

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Graph embeddings are of interest not only in topological graph theory but also within several areas of pure mathematics, physics and computing. They are a fundamental concept in combinatorics (products of permutations, Hopf algebra, chord diagrams), algebraic number theory (algebraic curves, Galois theory, Grothendiek's "dessins d'enfants", moduli spaces of curves and surfaces), knot theory (Vassiliev knot invariants) and theoretical physics (quantum field theory, string theory, Feynmann diagrams, Korteweg and de Vries equation), we refer to [29] for details. Every embedding of a graph can be described by a combinatorial map. Random maps with a given number of vertices have been the subject of much recent study. They have links with representation theory (conjugacy class products [11, 34]) and probability theory (the Brownian map, see [30] and the references therein). They also have applications in theoretical physics, via quantum gravity and matrix integrals, see [21, 45] for introductions to these fields. We will study the random maps obtained by randomly embedding a fixed graph or random graph. Despite these being natural models in random graph theory and probability theory, they have received less attention.

Existing work on random embeddings of graphs in surfaces is mostly concentrated on the notion of the *random genus* of a graph. By considering the uniform probability distribution on the set $\mathrm{Emb}(G)$ of all (equivalence classes of) 2-cell embeddings of a graph in (orientable) closed surfaces, we can speak of a random embedding and ask what is the expected value of its genus. The initial hope of using Monte Carlo methods on the configuration space of all 2-cell embeddings to compute the minimum genus of graphs [18, 20] quickly vanished as empirical simulations showed that, in many interesting cases, the average genus is very close to the maximum possible genus in $\mathrm{Emb}(G)$. The work of Gross and Rieper [18] also showed that there can be arbitrarily deep local minima for the genus that are not globally minimum. That result rules out traditional local-search algorithms. However it does not exclude search methods that have more significant random component, like the popular simulated annealing heuristic [42]. Our results show that for almost all graphs, starting with a random embedding we would be very far from a minimum with extremely high probability. Therefore, any heuristic with strong randomness will with high probability lead toward an embedding with only a few faces (and so of large genus). Hence, our work gives strong theoretical evidence that such methods are very unlikely to be successful. Of course, if we restrict inputs to a particular graph class such algorithms may still work. We conclude this paragraph with phrasing one of the main outcomes of our work; This paper provides a formal evidence that the Monte Carlo approach cannot work for approximating the minimum genus of graphs.

Unlike most previous works, we will not discuss the (average) genus but instead the (average) number of faces in random embeddings. Although the two variables are related linearly through Euler's formula, it turns out that the study of the number of faces yields a more appreciative view of certain phenomena that occur in this area.

1.2 State-of-the-art Random embeddings of two special families of graph are well understood. The first one is a bouquet of n loops (also called a *monopole*), which is the graph with a single vertex and n loops incident with the vertex. This family was first considered in a celebrated paper by Harer and Zagier [22] using representation theory. Several combinatorial proofs appeared later [7, 19, 23, 25, 43, 44]. By duality, the maps of the monopole with n loops correspond to unicellular maps [7] with n edges. The second well-studied case is the n-*dipole*, a two-vertex graph with n edges joining the two vertices; see [1, 8, 9, 12, 24, 25, 28, 35]. A more recent case gives an extension to the "multipoles" [4] using a result of Stanley [41]. Random embeddings in all these cases are in bijective correspondence with products of permutations in two conjugacy classes. A notable generalization of these cases appears in a paper by Chmutov and Pittel [11]. Another well-studied case includes "linear" graph families, obtained from a fixed small graph H by joining n copies of H in a path-like way, see [17, 39] and references therein.

Here we discuss random graphs, including dense cases. One special case, which is of particular importance, is that of complete graphs. Looking at the small values of n, K_3 has only one embedding, which has two faces. It is easy to see that K_4 has two embeddings of genus 0 (with four faces) and all other embeddings have genus 1 and two faces. A brute force calculation using a computer gives the numbers for K_5 and K_6. They are collected in Table 1. The genus distribution of K_7 has been computed only recently [3, 37] and there is no data for larger number of vertices. The computed numbers for K_n show that for $n \leq 7$ most embeddings have a small number of faces. The results of this paper show that, similarly to the small cases, most embeddings of any K_n will have large genus and the average number of faces is not only subquadratic but it is actually proportional to $\ln n$.[2] This is a somewhat surprising outcome, because the complete graph K_n has many embeddings with $\Theta(n^2)$ faces. In fact,

[1]This value was computed explicitly in [37, Table 3.1].
[2]We use $\ln n$ to denote the natural logarithm.

n	3	4	5	6
$\mathrm{emb}(K_n)$	1	2^4	6^5	24^6
$g=0$	1	2	0	0
$g=1$	0	14	462	1,800
$g=2$	0	0	4,974	654,576
$g=3$	0	0	2,340	24,613,800
$g=4$	0	0	0	124,250,208
$g=5$	0	0	0	41,582,592
$\mathbb{E}(g)$	0	0.875	2.24	4.082

(a) Genus distribution

n	3	4	5	6	7
$\mathrm{emb}(K_n)$	1	2^4	6^5	24^6	
$F=1$	0	0	2,340	41,582,592	
$F=2$	1	14	0	0	
$F=3$	0	0	4,974	124,250,208	
$F=4$	0	2	0	0	
$F=5$	0	0	462	24,613,800	
$F=6$	0	0	0	0	
$F=7$	0	0	0	654,576	
$F=8$	0	0	0	0	
$F=9$	0	0	0	1,800	
$\mathbb{E}(F)$	2	2.25	2.517	2.836	3.1265[1]
$\approx 2\ln n$	2.2	2.77	3.22	3.58	3.89

(b) Face distribution

Table 1: Data obtained by exhaustive computation concerning K_n for $n \leq 6$

it was proved by Grannell and Knor [16] (see also [14] and [15]) that for infinitely many values of n there is a constant $c > 0$ such that the number of embeddings with precisely $\frac{1}{3}n(n-1)$ faces is at least n^{cn^2}. All these embeddings are triangular (all faces are triangles) and thus of minimum possible genus. When we compare this result with the fact that

$$|\operatorname{Emb}(K_n)| = ((n-2)!)^n = n^{\Theta(n^2)},$$

we see that there is huge abundance of embeddings of K_n with many more than logarithmically many faces.

Stahl [38] introduced the notion of *permutation-partition pairs* with which he was able to describe partially fixed rotation systems. Through the linearity of expectation these became a powerful tool to analyze what happens in average. In particular, he was able to prove that the expected number of faces in embeddings of complete graphs is much lower than quadratic.

Theorem 1.1. (Stahl [40, Corollary 2.3]) *The expected number of faces in a random embedding of the complete graph K_n is at most $n + \ln n$.*

Computer simulations show that even the bound given in Theorem 1.1 is too high. In fact, Mauk and Stahl conjectured the following.

Conjecture 1.1. (Mauk and Stahl [32, page 289]) *The expected number of faces in a random embedding of the complete graph K_n is at most $2\ln n + O(1)$.*

For general graphs, a slightly weaker bound than that of Theorem 1.1 was derived by Stahl using the same approach as in [40]; it had appeared in [39] a couple of years earlier.

Theorem 1.2. (Stahl [39, Theorem 1]) *The expected number of faces in a random embedding of any n-vertex graph is at most $n \ln n$.*

The $n \ln n$ bound of Stahl was improved only recently. Campion Loth, Halasz, Masařík, Mohar, and Šámal [4] conjectured that the bound should be linear, which was then proved in [6].

Theorem 1.3. (Campion Loth and Mohar [6, Theorem 3]) *The expected number of faces in a random embedding of any graph is at most $\frac{\pi^2}{6}n$.*

The bound of Theorem 1.3 is essentially best possible as there are n-vertex graphs whose expected number of faces is $\frac{1}{3}n + 1$, see [6].

1.3 Our results

1.3 Our results The first main contribution of this paper is the proof of Conjecture 1.1 with a slightly worse multiplicative factor.

THEOREM 1.4. *Let $n \geq 1$ be an integer and let $F(n)$ be the random variable whose value is the number of faces in a random embedding of the complete graph K_n. The expected value of $F(n)$ is at most $10 \ln n + 2$. For n sufficiently large ($n \geq e^{e^{16}}$) the multiplicative constant is even better, namely:*

$$\mathbb{E}[F(n)] \leq 3.65 \ln n.$$

We complement our upper bound with a lower bound showing that our result is tight up to the multiplicative factor.

THEOREM 1.5. *For all positive integers n, we have*

$$\mathbb{E}[F(n)] > \tfrac{1}{2} \ln(n) - 2.$$

In order to prove Theorem 1.4, we split the proof into ranges based on the value of n and use a different approach for each range. In fact, we provide two theoretical upper bounds using a close examination of slightly different random processes. The first one is easier to prove, but it gives an asymptotically inferior bound. However, it is useful for small values of n. In the bound, we use the harmonic numbers $H_k := \sum_{j=1}^{k} \frac{1}{k}$, whose value is approximately equal to $\ln n$.

THEOREM 1.6. *Let $n \geq 10$ be an integer. Then*

$$\mathbb{E}[F(n)] < H_{n-3} H_{n-2}.$$

Note that proof of Theorem 1.6 also works for $n \geq 4$, but yields a slightly worse bound (see Equation (3.1)), which we have not stated above. Moreover, we used Equation (3.1) to estimate values for $n \leq 242$ using computer and this implies Theorem 1.4 ($E[F(n)] \leq 5 \ln n + 5$) for this range; see [5, Section 5] for the details.

The next theorem is our core result that implies Theorem 1.4 for $n > 40748$.

THEOREM 1.7. *For $n \geq e^{e^{16}}$, $\mathbb{E}[F(n)] \leq 3.65 \ln(n)$. For $n \geq e^{30}$, $\mathbb{E}[F(n)] \leq 5 \ln(n)$. For $e^{10.6} \approx 40748 \leq n < e^{30}$, $\mathbb{E}[F(n)] \leq 10 \ln(n) + 2$.*

For small values of $243 \leq n \leq 40748$ we used a computer-assisted proof which is based on our general estimates given in the proof of Theorem 1.7 combined with pre-computed bounds for smaller values of n and Markov inequality. We will give more details on our computation in [5, Section 5]. We summarize the results of computer-calculated upper bounds in the following proposition. Note that having a small additive constant for small values of n helps us to keep smaller additive constants for middle values of n as our proof is inductive.

PROPOSITION 1.1. *For $1 \leq n \leq 40748$, $\mathbb{E}[F(n)] \leq 5 \ln(n) + 5$.*

In summary, the proofs of the above results for complete graphs are relatively long. A "log-square" improvement of Stahl's linear bound is not that hard, but the $O(\ln n)$ bound appears challenging and shows all difficulties that arise for more general dense graph classes.

In the second part of the paper, we turn to more general random graph families. Let $F(n, p)$ be the random variable for the number of faces in a random embedding of a random graph in $G(n, p)$. We will first show a bound on the expectation of this variable which holds for any value of p.

THEOREM 1.8. ($\spadesuit^3$) *Let n be a positive integer and $p \in (0, 1]$ ($p = p(n)$). Then we have:*

$$\mathbb{E}[F(n, p)] \leq H_n^2 + 1/p.$$

Theorem 1.8 gives a "log-square" general bound which can be improved in the sparse regime as well as in the dense regime (for multigraphs). First, we state a general result for random embedding of random maps with fixed degree sequence. In other words, we will investigate random embeddings of random multigraphs possibly with loops sampled uniformly out of multigraphs with the same fixed degree sequence. Some results of this flavor have been obtained earlier in the setup of "random chord diagrams", see [10, 31].

THEOREM 1.9. (♠) *Let* $\mathbf{d} = (t_1, t_2, \dots, t_n)$ *be a degree sequence for an* n*-vertex multigraph (possibly with loops) where* $t_i \geq 2$ *for all* i*. Let* $\mathbb{E}[F_{\mathbf{d}}]$ *be the average number of faces in a random embedding of a random multigraph with degree sequence* $\mathbf{d}$*. Then* $\mathbb{E}[F_{\mathbf{d}}] = \Theta(\ln n)$*.*

However, we are mostly interested in simple graphs. For larger degree sequences, the majority of random embeddings generated in the model of Chmutov and Pittel [10, 31] will not be simple. Therefore, we will be focusing on degree sequences with bounded parts while we allow n to grow to infinity. Given a degree sequence $\mathbf{d} = (t_1, t_2, \dots, t_n)$, let

$$m_{\mathbf{d}} = \frac{1}{2}\sum_i t_i \quad \text{and} \quad \lambda_{\mathbf{d}} := \frac{1}{2m_{\mathbf{d}}}\sum_{i=1}^{n}\binom{t_i}{2}.$$

Janson [27] showed that a random multigraph with degree sequence $\mathbf{d}$ is asymptotically almost surely *not* simple unless $\lambda_{\mathbf{d}} = O(1)$. This means, for example, that the probability of a d-regular multigraph on n vertices being simple is bounded away from 0 only if d is constant (while n grows arbitrarily). Restricting our attention to the case where vertex degrees are bounded by an absolute constant, Janson's result tells us that simple graphs make up a nontrivial fraction of all multigraphs with a given degree sequence. In fact, this special case of Janson's result was obtained over 30 years earlier by Bender and Canfield [2]. We prove that, in the case of random *simple* graphs with constant vertex degrees, we preserve logarithmic bounds on the expected number of faces.

THEOREM 1.10. (♠) *Let* $d \geq 2$ *be a constant,* $\varepsilon > 0$ *(a constant within* Θ *depends on* ε*), and let* $\mathbf{d} = (t_1, t_2, \dots, t_n)$ *be a degree sequence for some* n*-vertex simple graph with* $2 \leq t_i \leq d$ *for all* i*, and such that* $m_{\mathbf{d}} \geq (1+\varepsilon)n$*. Let* $\mathbb{E}[F^s_{\mathbf{d}}]$ *be the average number of faces in a random embedding of a random simple graph with degree sequence* $\mathbf{d}$*. Then* $\mathbb{E}[F^s_{\mathbf{d}}] = \Theta_\varepsilon(\ln n)$*.*

In the light of the above theorems and our Monte Carlo experiments, we conjecture that a logarithmic upper bound should be achievable for any usual model of random graphs. However, extending our proof of Theorem 1.4 to arbitrary random graphs seems to require further ideas.

CONJECTURE 1.2. *Let* $p = p(n)$ *be the probabillity of edges in* $G(n,p)$*. The expected number of faces in a random embedding of a random graph* $G \in G(n,p)$ *is*

$$(1 + o(1))\ln(pn^2).$$

We refer to Section 6 for further discussion on conjectures and open problems that are motivated by our results.

Structure of the paper. Before we dive into proofs we will present our common strategy and formalization used in Theorems 1.6 and 1.7 in Section 2. First, we present the easier proof of Theorem 1.6 in Section 3. Our main result (Theorem 1.7) on complete graphs can be found in Section 4 with full details avalable in [5]. We conclude the complete graph sections with a short proof of our lower bound (Theorem 1.5) in Section 5. In Section 6, we discuss conjectures and open problems.

1.4 Preliminaries

Combinatorial maps. To describe 2-cell embeddings of graphs we need a formal definition of a map. A *combinatorial map* (as introduced in [26, 36]) is a triple $M = (D, R, L)$ where

- D is an abstract set of *darts*;
- R is a permutation on the symbols in D;
- L is a fixed point free involution on the symbols in D.

Combinatorial maps are in bijective correspondence with 2-cell embeddings of graphs on oriented surfaces, up to orientation-preserving homeomorphisms. See [33, Theorem 3.2.4] for a proof. We give details of this correspondence. Let $G = (V, E)$ be a graph on n vertices, where $V = \{v_1, \dots, v_n\}$.

- For $i \in [n]$, let D_i be the set of all pairs (v_i, e) where e is an edge incident with v_i. Note that $|D_i| = t_i$ is the degree of v_i. Let $D = D_1 \cup \cdots \cup D_n$ be the set of all darts.

- For each $i \in [n]$, we let R_i be a unicyclic permutation of darts in D_i, in clockwise order as they emanate from v_i on the surface. So, $R_i(d)$ is the dart following d in the clockwise order given by R_i, and conversely $R_i^{-1}(d)$ is the dart preceding d in this cyclic order. We let $R = R_1 R_2 \cdots R_n$, and call R a *rotation system.*

- We let L be a permutation of D consisting of 2-cycles swapping (v_i, e) with (v_j, e) for each edge $e = ij$. We call L an *edge scheme.*

- The cycles of the permutation $R \circ L$ give the faces of the embedding.

Conversely, starting with a combinatorial map $M = (D, R, L)$, we define the graph whose vertices are the cycles of R, and whose edges are the 2-cycles of L.

Random embeddings. Fix an arbitrary edge scheme L. It is well known that all the 2-cell embeddings of G, up to homeomorphism, are given by the set of all (D, R, L) over all rotation systems R. We call an embedding chosen uniformly at random from the set of these maps a *random embedding* of G.

Now fix some rotation system R. Intuitively, given G we know what vertices are connected by an edge, say $uv \in E(G)$, but within the dart model, we do not know what particular dart incident with u connects to a particular dart of v. Hence, we argue that we can model a random embedding of G just by picking what darts form the edges uniformly at random. Indeed, a simple counting argument shows that for G with degree sequence $t_1, \ldots, t_n$, there are $t_1!t_2! \ldots t_n!$ possible edge schemes. Moreover, each embedding of G is given by $t_1 t_2 \cdots t_n$ different edge schemes. In particular, each embedding is given by the same number of edge schemes. Therefore we may also obtain a random embedding of G by fixing some rotation system R and picking a uniform at random edge scheme. This is the model we will use in Sections 3.

Thirdly, we may vary both the local rotation and the edge scheme. Picking a uniform at random rotation system and edge scheme also gives a random embedding of G. This is the model we will use in Section 4.

Partial maps and temporary faces. Our proofs will involve building up a map step by step. Therefore we will need a notion of a partially constructed map. A *partial map* is defined in the same way as a map (D, R, L), except L need not be fixed point free. We define the darts that are in 2-cycles in L as *paired darts* and the darts that are fixed points in L as *unpaired darts.*

The *faces* of the implied embedding of a map $M = (D, R, L)$ are given by the orbits of $R \circ L$. One of our main interests in this paper will be the number of faces. In a partial map, each cycle in $R \circ L$ may contain some number of unpaired darts and/or paired darts. For a partial map (D, R, L), a cycle of $R \circ L$ is a *completed face* if it contains only paired darts, and a *temporary face* if it contains at least one unpaired dart. In particular, we say a temporary face is *k-open* if it contains precisely k unpaired darts. We say that a temporary face f is *strongly 2-open* if f is 2-open and the two unpaired darts in f are incident with different vertices.

Our proofs are often stated in terms of *facial walks.* For a completed face, this is simply a walk around the boundary of the face. For temporary face, this is a walk where we travel along the paired darts which make up edges, but walk through any unpaired dart. Let f be a k-open face and let $d_1, d_2, \ldots, d_k$ be the unpaired darts that belongs to f in their anti-clockwise order of appearance on a facial walk around f. For each i $(1 \le i \le k)$, we call the segment of a facial walk around f from d_i to d_{i+1} the *partial facial walk* (*partial face*) with initial dart d_i and ending dart d_{i+1}. (We also say that this partial face *leads from d_i to d_{i+1}*. Note that each unpaired dart is the initial dart for precisely one partial face and is also the ending dart of precisely one partial face.

Now, we introduce a theorem describing fast convergence of H_n. Note that the lower bound works also for $n = 0$ since $H_0 = 0 \ge \ln\left(\frac{1}{2}\right) + \gamma + \frac{1}{24}$.

THEOREM 1.11. (FAST CONVERGENCE OF H_n [13]) *For every $n \ge 1$, we have*

$$H_n = \ln\left(n + \tfrac{1}{2}\right) + \gamma + \varepsilon_n,$$

where $\frac{1}{24(n+1)^2} \le \varepsilon_n \le \frac{1}{24n^2}$ and $\gamma \approx 0.57721$ is the Euler–Mascheroni constant.

2 Our proof strategy for complete graphs

We give two proofs of a bound on $\mathbb{E}[F(n)]$. One gives an asymptotically worse bound, but will be useful to give the best estimates for small values of n. The other one is more involved and requires rather tedious computation. Here we present an intuition on both proofs and introduce a bit more terminology.

Log-square bound. For this proof, we will fix an arbitrary rotation system and pick a uniform at random edge scheme. We will work with a random process that builds a random edge scheme step by step. First, we order the vertices of a graph G arbitrarily. We represent the ordering as $v_1, v_2, \ldots, v_n$, and we process vertices one by one, starting with v_n. When processing a vertex v_k, since we fixed a rotation system, the cyclic rotation of darts in D_k is fixed. We process the darts incident with v_k in this fixed order. At each step we either keep this dart unpaired or pick another random dart to pair this dart with to make a 2-cycle in L, as defined precisely in Random Process A. An analysis of this process in Section 3 will give Theorem 1.6.

Logarithmic bound. In this proof, we use a more refined random process to generate a random rotation system, and a random edge scheme. We then conclude by rather complicated computation. In a similar manner to the previous description, we process vertices one at at time, and process darts one at a time at each vertex.

When we process v_k, we refer to it as *step k*. For each $k \in [n-2]$, we define the following terminology. Let $V^{\uparrow}$ be vertices $v_n, \ldots, v_{k+1}$ and $D_{V^{\uparrow}}$ be set of their darts. Recall that dart d is unpaired if $L(d)$ is undefined. Now, we make the following random choice. For each $i > k$ we choose uniformly at random an unpaired dart $d_i \in D_i$ and we define $L(d_i) := d$ for some unpaired dart $d \in D_k$. We call all such newly paired darts *active* for this step. Observe that $k-1$ darts remain unpaired at vertex v_k in this step.

We then study how many of various types of active darts we expect to obtain from this random choice. Based on this, we randomly build a rotation system at v_k. We do this step by step: we fix some processing order of the darts in D_k. Then for each dart d in this order, we randomly choose a value of $R(d)$. This will be defined precisely as Random Process B. Analysing the probability of adding a completed face to the embedding when assigning each value of $R(d)$ will give the proof of Theorem 1.7.

3 Log-square bound—proof of Theorem 1.6

We start by proving Theorem 1.6.

THEOREM 1.6. *Let $n \geq 10$ be an integer. Then*

$$\mathbb{E}[F(n)] < H_{n-3}H_{n-2}.$$

We will use a similar approach for the proof of Theorem 1.8 in [5, Section 7]. Refer to Figure 1 for an example of this random process.

Random process A.

1. Order the vertices of the graph $v_n, \ldots, v_1$ arbitrarily and process the vertices in this order.
2. Start with vertices v_n and v_{n-1}. They belong to one temporary face and no face has been closed so far.
3. Consider vertex v_k for $k \in [n-2]$. Label the darts of D_k as $\{d_1, \ldots, d_{n-1}\}$ arbitrarily. We define R_k as this cyclic order, that is $R_k(d_i) = d_{i+1}$ (except $R_k(d_{n-1}) = d_1$). Let $C_k := \{n, n-1, \ldots, k+1, u, u, \ldots, u\}$ where there are $k-1$ copies of the symbol u representing that the dart choosing u remains unpaired. This is the multi-set of choices of where the darts may lead at the end of this step.
   (a) Process darts in D_k in order $d_1, d_2, \ldots, d_{n-1}$. If $k > 1$, give d_1 the label u, remove one copy of u from C_k, and proceed processing d_2. If $k = 1$, start by processing d_1.
   (b) Consider the dart d_ℓ which is next in the order. **Random choice 1a:** Pick a symbol from the set C_k uniformly at random, then remove this choice from C_k.
      - Case 1: *The choice was some $i \geq k+1$.* **Random choice 1b:** Then pick an unpaired dart d' uniformly at random from those at v_i. Then add the transposition (d', d_ℓ) to the permutation L.
      - Case 2: *The choice was some u.* Then leave dart d_ℓ unpaired.

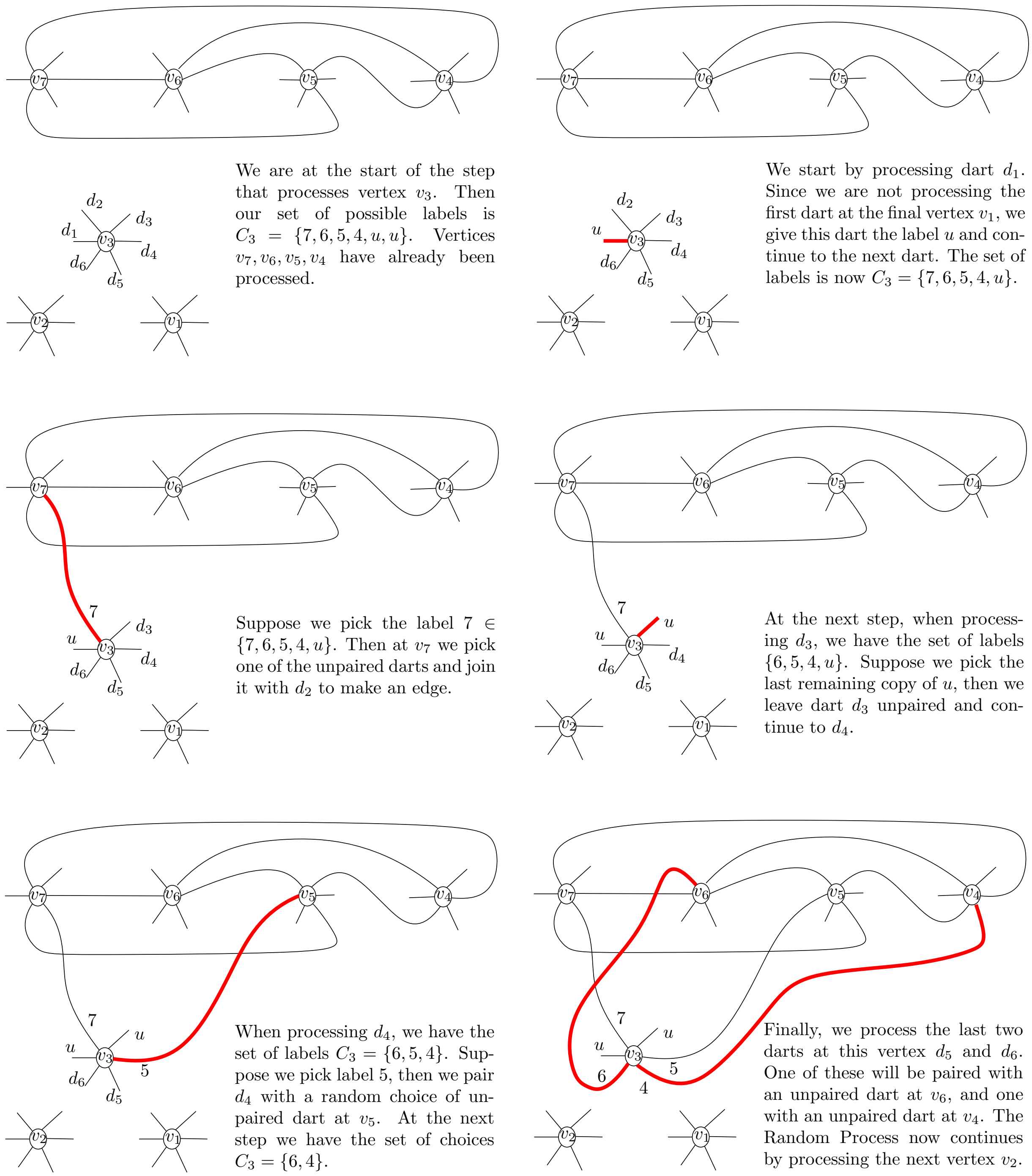

Figure 1: An example of Random Process A, processing vertex v_3.

Continue to the next dart in the order.

Continue to the next vertex in the order. ⌟

For each value of $k \leq n-2$, let F_k ($F_k = F_k(n)$) be the number of faces completed at step k. By this, we mean the facial walks that contain v_k and no vertex v_j with $j < k$. They were completed at step k and have stayed unchanged until the end of the process. We need an upper bound on $\mathbb{E}[F_k]$. By linearity of expectation, we have that $\mathbb{E}[F(n)] = \sum_{k=1}^{n-2} \mathbb{E}[F_k(n)]$.

Suppose we are processing the dart d_ℓ at step k. Recall that d_ℓ is contained in two partial faces: one starting at some dart d and ending at d_ℓ, and one starting at d_ℓ and ending at some dart d'. We complete a face at this step if and only if we pair d_ℓ with dart d or d'. The dart d' is an unpaired dart incident with v_k with a single exception when $k = 1$ and $\ell = n-1$. So pairing d_ℓ with d' can not completed a face unless we have this exception. We have two cases:

Case 1: $\ell = 1$*, or the previously processed dart* $d_{\ell-1}$ *was chosen to be unpaired:* Then both darts d and d' are incident with vertex v_k, so we cannot pair with them. Therefore, we cannot have completed a face when processing d_ℓ.

Case 2: $d_{\ell-1}$ *is paired:* See Figure 2 for an example of this analysis. We complete a face at this step if and only if we pair d_ℓ with d, where d is the dart at the start of the partial face ending at d_ℓ. The probability we choose d_ℓ to lead to vertex v_i, for $i > k$, is at most $\frac{1}{n-\ell}$ as we have already chosen $\ell - 1$ vertices in Random choice 1a. The probability that we choose dart d (and not another unused dart at v_i) to connect with d_ℓ is $\frac{1}{k}$ as there are k unpaired darts incident vertex v_i to choose from in Random choice 1b. Therefore, the probability that we complete the face is at most $\frac{1}{k(n-\ell)}$.

Case 3: $k = 1$*:* When processing d_{n-1}, the dart d' at the end of the partial face starting at d_{n-1} is not at v_1. Therefore, we can close two faces at this step.

Assume now $k > 1$. Each dart (except for d_1) has probability $\frac{n-k}{n-2}$ of being paired (as d_1 is unpaired). Thus a dart d_ℓ ($\ell \geq 3$) has the same probability $\frac{n-k}{n-2}$ of being Case 2. Therefore, the probability that we close a face by pairing up d_ℓ is at most $\frac{n-k}{n-2} \cdot \frac{1}{k(n-\ell)}$.

For $k = 1$, all edges are connected to $V^\uparrow$, thus the probability of closing a face by d_ℓ (for $\ell \geq 2$ now) is $\frac{1}{n-\ell}$. Moreover, the last dart d_{n-1} can close two faces as described in Case 3.

Summing over all values of ℓ we get for $k \geq 2$ and $n \geq 4$

$$\mathbb{E}[F_k] \leq \sum_{\ell=3}^{n-1} \frac{n-k}{n-2} \cdot \frac{1}{k(n-\ell)} = \frac{n-k}{k(n-2)} \cdot H_{n-3}.$$

Also,

$$\mathbb{E}[F_1] \leq 1 + \sum_{\ell=2}^{n-1} \frac{1}{n-\ell} = 1 + H_{n-2}.$$

Summing over all steps k assuming $n \geq 4$ (apart from the last line) we obtain:

$$\begin{aligned}
\mathbb{E}[F] &= \mathbb{E}[F_1] + \sum_{k=2}^{n-2} \mathbb{E}[F_k] \\
&\leq 1 + H_{n-2} + \sum_{k=2}^{n-2} \frac{n-k}{k(n-2)} H_{n-3} \\
&= 1 + H_{n-2} + \frac{1}{n-2}(nH_{n-2} - n + 2)H_{n-3} - \frac{n-1}{n-2} H_{n-3} \\
&= 1 + H_{n-2} + \frac{n}{n-2} H_{n-3}(H_{n-2} - 1) - \frac{n-3}{n-2} H_{n-3} \qquad (3.1) \\
&< H_{n-3} H_{n-2}. \qquad \text{(for } n \geq 10)
\end{aligned}$$

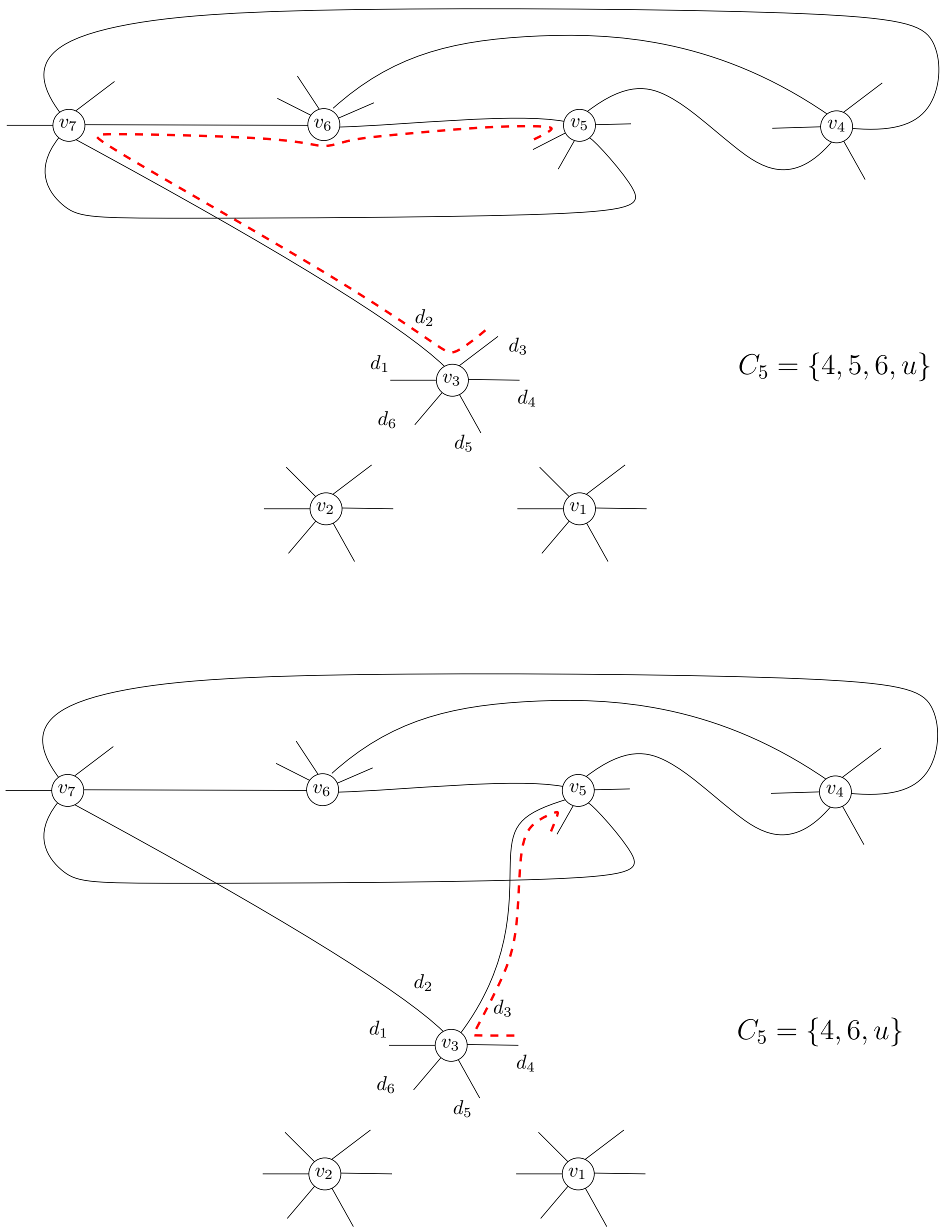

Figure 2: The upper diagram shows the step of Random Process A where we are processing dart d_3 at vertex v_3. The partial facial walk is traced in dotted red line, showing the only dart for which pairing with makes a completed face. At the next step, the only dart for which pairing with makes a completed face is at vertex v_5. However, we have already added the edge from v_3 to v_5, so 3 is not a valid choice of a label at this step. Therefore, we cannot add a completed face at this step.

4 Logarithmic bound—proof of Theorem 1.7

THEOREM 1.7. *For* $n \geq e^{e^{16}}$, $\mathbb{E}[F(n)] \leq 3.65\ln(n)$. *For* $n \geq e^{30}$, $\mathbb{E}[F(n)] \leq 5\ln(n)$. *For* $e^{10.6} \approx 40748 \leq n < e^{30}$, $\mathbb{E}[F(n)] \leq 10\ln(n) + 2$.

We first introduce more notation that will be needed in the proof. We look more carefully at step k. At this step the walks in $R \circ L$ can be split into two categories building on notation defined in Section 1.4:

1. *Completed faces:* cycles of $R \circ L$. Those are closed walks that corresponds to 0-open faces which will not change any more, and

2. *Candidate walks:* those are partial faces that originates at an unpaired dart d_s and lead to an unpaired dart d_e (possibly $d_s = d_e$).

For each vertex in $V^{\uparrow}$, we will pick an *active dart* randomly from the set of all unpaired darts incident with this vertex. Observe that if a partial face starts with a dart d_s and ends with d_e, then it can complete a face in step k only if both d_s and d_e become active. We call such walks *active* in step k. We further partition the active walks into

(1) Those for which $d_s = d_e$. Observe that such are necessarily 1-open faces and so we refer to them as *1-open active faces*, and

(2) All other active walks (i.e., $d_s \neq d_e$), which we refer to as *potential faces*.

An active dart $d \in D_k$ is called *1-open* if $L(d)$ is the dart incident with some 1-open face. An active dart $d \in D_k$ is called *potential* if $L(d)$ is incident with some potential face. We will give more intuition on our terminology. We will show that under certain circumstances, only potential faces may complete a face. Therefore, we call unpaired darts in D_k together with darts that do not take part in any active walk *non-contributing*. Let PF_k be a random variable representing the number of potential faces and O_k be a random variable representing the number of 1-open active faces in step k, after active darts were chosen. Let F_k denote the total number of completed faces added during step k.

We now describe our random procedure in detail. We refer to Figure 3 for an example of this random process.

Random process B.

1. Label the vertices arbitrarily as $v_n, \ldots, v_1$ and process them in that order.

2. Start with vertex v_n, and fix a uniform at random full cycle R_n. This vertex is incident with $n-1$ unpaired darts.

3. Consider vertex v_k for $k \in [n-1]$, starting with $n-1$.

   (a) **Random choice 1:** For each vertex in $V^{\uparrow}$ we choose uniformly at random one out of k unpaired darts to lead to v_k and update L appropriately. The chosen darts are said to be the *active darts* at step k.

   (b) We treat D_k as an unordered set, and build a local rotation R_k by processing the darts in a special order σ_k given by the type of walk the dart describes. Each time we fix $R_k(d)$ for the processed dart d. We define σ_k as follows:

      i. First, process 1-open darts in arbitrary order.

      ii. Next, potential darts follow in arbitrary order.

      iii. Last, non-contributing darts are processed, again in arbitrary order.

   (c) **Random choice 2:** For each $d \in D_k$ in order σ_k we choose uniformly at random one dart d' among all possible options (those that do not violate the property that R_k will define a single cycle eventually) and we set $R_k(d) := d'$. ⌟

Now, we define a function q, which will form an upper bound for the contribution of vertex v_k to the expected number of faces. The function is defined as follows. (Note that $H_0 = 0$.)

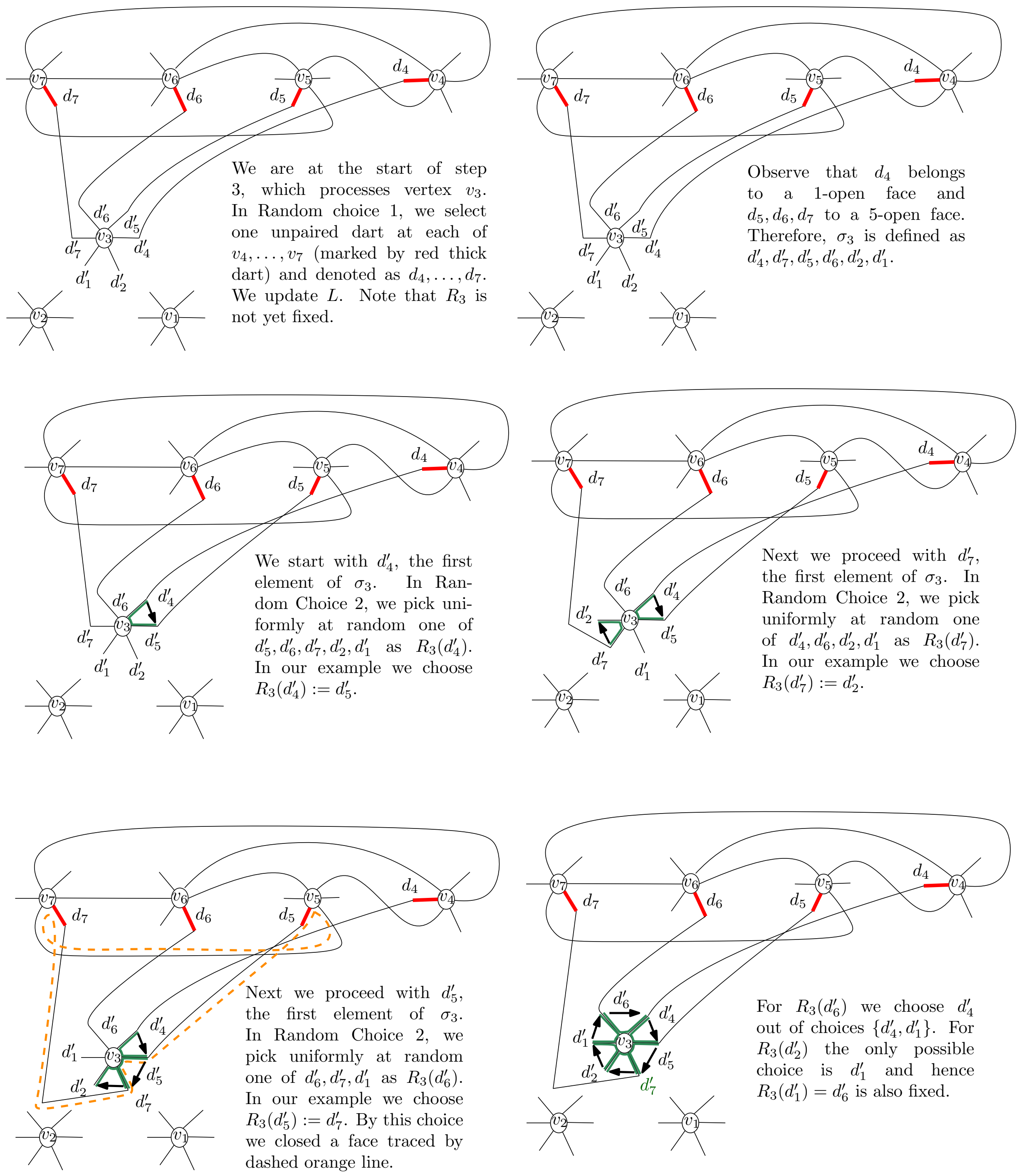

Figure 3: An example of Random Process B, processing vertex v_3 to obtain R_3. At the end of this step, the darts d'_1, d'_2 remain unpaired. It is not decided which one will go to v_1 and which one will go to v_2.

Definition 4.1. *If $1 \le t < n$ and $0 \le \xi < n-1-t$, then*

$$(4.2)\qquad q(\xi, t) \coloneqq \sum_{i=1}^{t} \frac{1}{n-\xi-i-1} = H_{n-\xi-2} - H_{n-\xi-t-2}.$$

If $\xi + t = n-1$ then

$$(4.3)\qquad q(\xi, t) \coloneqq \sum_{i=1}^{t-1} \frac{1}{n-\xi-i-1} + 1 = H_{n-\xi-2} + 1.$$

It is easy to observe the following fact about the function q:

Observation 4.1. *Let $a \ge 1$, $1 \le t+a < n$, and $0 \le \xi - a < n-1-t-a$. Then*

$$q(\xi, t) \le q(\xi - a, t + a).$$

Now, we state the crucial lemma that is a starting point of the upper bound computation.

Lemma 4.1. (♠) *Given $PF_k = t$ and $O_k = \xi$, the average number of faces completed at vertex v_k is at most $q(\xi, t)$. In other words, $\mathbb{E}[F_k | PF_k = t, O_k = \xi] \le q(\xi, t)$.*

Note that $O_k + PF_k$ is never larger than $n-1$ and therefore the value $q(\xi, t)$ is well-defined. Observe that $O_k + PF_k = n-1$ if and only if $k = 1$ as there are exactly $n-k$ edges between v_k and $V^{\uparrow}$.

We define one more random variable. Let T_{n-k} represent the number of temporary faces in $G[V^{\uparrow}]$ in step k (before vertex v_k is added). Note that $E[T_n]$ is, in other words, an average number of faces of K_n. Hence, the following lemma is the first step in the proof of the main theorem. The rest of the proof will provide an involved analysis of the right-hand side of Inequality (4.4).

Lemma 4.2. *Let $n \ge 3$ and let F, PF_k, O_k be random variables as defined earlier. Then we have:*

$$(4.4)\qquad \mathbb{E}[F] = \mathbb{E}[T_n] \le \sum_{k=1}^{n-2} \mathbb{E}[q(O_k, PF_k)] = \sum_{k=1}^{n-2} \sum_{i=1}^{n-k} \sum_{j=0}^{n-k-i} q(j, i) \cdot \Pr[O_k = j \wedge PF_k = i].$$

Proof. The equalities in (4.4) are clear, so we will only argue about the inequality. We execute Random process B as defined above. For the first two vertices v_n and v_{n-1} in the order, all choices are isomorphic. We process each other vertex as described in part 3 of the process description. Hence, the contribution of a single vertex is upper-bounded by Lemma 4.1 □

Let $1/2 < \nu < 1$ be a constant and $\overline{\nu} \coloneqq 1 - \nu$. We will fix this value later on for different ranges of n in order to optimise our bound. We split the above triple sum (Equation (4.4) in Lemma 4.2) into several parts:

- S_1 will contain the terms where $k = 1$.
- S_2 will contain the terms where $j < \overline{\nu} n$ and $i < \frac{n-k}{k}$.
- S_3 will contain the terms where $j < \overline{\nu} n$ and $i \ge \frac{n-k}{k}$.
- S_4 will contain the terms where $j \ge \overline{\nu} n$.

There $\gamma \approx 0.57721$ denote Euler-Mascheroni constant. We now define S_1, S_2, S_3, and S_4. We will also state the bounds which we derive for each portion of the sum in the forthcoming subsections.

$$(4.5)\qquad S_1 \coloneqq \sum_{i=1}^{n-1} \sum_{j=0}^{n-1-i} q(j, i) \cdot \Pr[O_1 = j \wedge PF_1 = i] \le H_{n-2} + 1 \le \ln(n) + \gamma + 1.$$

For the rest, we first take the terms for which $O_k < \overline{\nu}n$. Let $b = b(n,k,i) \coloneqq \min(n-k-i, \lceil \overline{\nu}n \rceil - 1)$. When writing down the terms for S_2, we used the fact that these terms do not occur if $\frac{n-k}{k} \le 1$. Thus we have the summation range for k only between 2 and $n/2$.

$$
\begin{aligned}
S_2 \coloneqq & \sum_{k=2}^{n/2} \sum_{i=1}^{\lceil \frac{n-k}{k} \rceil - 1} \sum_{j=0}^{b} q(j,i) \cdot \Pr[O_k = j \wedge PF_k = i] \\
\le & \frac{1}{\nu} \ln(n) + \ln\left(\frac{\nu n - 3/2}{\nu n - 1/2 - \frac{n}{2}}\right) + \frac{1}{\nu}\left(\ln(\nu/2) - \ln(5\nu/2 - 1)\right) .
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
S_3 \coloneqq & \sum_{k=2}^{n-2} \sum_{i=\lceil \frac{n-k}{k} \rceil}^{n-k} \sum_{j=0}^{b} q(j,i) \cdot \Pr[O_k = j \wedge PF_k = i] \\
\le & \ln(2\nu n) \frac{\frac{\pi^2}{6} - 1}{\nu^2} \left(1 + \frac{4}{\nu n - 2}\right) + 1.67 \ln n + 5 + \frac{2n}{\nu n - 5/2}.
\end{aligned} \tag{4.7}
$$

In case $n \ge e^{e^{16}}$ and $\nu \ge \frac{999}{1000}$, we have a stronger estimate:

$$
S_3 \le 1.6474 \ln n - 9. \tag{4.8}
$$

Finally, we take the remaining case where $O_k \ge \overline{\nu}n$. The corresponding inequality involves an auxiliary (real) parameter $\mu \in [1,3]$, and an integer $\aleph_m \in \mathbb{Z}$ such that $\mathbb{E}[F(m)] \le 5\ln(m) + \aleph_m$ for all $2 \le m < n$. We denote $\aleph_b^a \coloneqq \max_{b<i<a} \aleph_i$ for $0 < b < a$.

$$
S_4 \coloneqq \sum_{k=2}^{n-2} \sum_{i=1}^{n-k} \sum_{j=\lceil \overline{\nu}n \rceil}^{n-k-i} q(j,i) \cdot \Pr[O_k = j \wedge PF_k = i] \tag{4.9}
$$

$$
< \nu n \ln(\nu n) e^{\frac{-n\overline{\nu}^2}{2}} + \frac{\nu \ln(\nu n)\left(5\ln n + \aleph_{\lceil \overline{\nu}n \rceil}^{n - \lceil \frac{2}{\nu} ln^{\mu}(n) \rceil}\right)}{\ln^{\mu}(n)} + \frac{2\ln^{\mu}(n)\ln(\nu n)\left(5\ln n + \aleph_{n - \lceil \frac{2}{\nu} ln^{\mu}(n) \rceil + 1}^{n-2}\right)}{\overline{\nu}^2 n} \tag{4.10}
$$

Lemma 4.2 together with the above analysis reformulates Theorem 1.7 as the following inductive theorem. The base case of the induction is computed using the computer analysis formulated as Theorem 1.1. Note that it is sufficient to assume $n \ge 243$ for the next theorem as the smaller values follow from Theorem 1.6 via computer-evaluation which is described later in [5, Section 5]. Observe that if we do not aim for the best multiplicative constant we can use our $\ln^2 n$ upper bound (Theorem 1.6) in the place of the inductive argument. However, it would not be sufficient to use there, for example, the previously known linear bound.

Theorem 4.1. (♠) *Let $n \ge 243$ be an integer. For $3 \le m < n$, suppose that $\mathbb{E}[F(m)] \le 5\ln(m) + \aleph_m$. Then we have:*

$$
\mathbb{E}[F(n)] \le S_1 + S_2 + S_3 + S_4
$$

where S_1, S_2, S_3, S_4 are defined above in Equations (4.5), (4.6), (4.7), and (4.9).

Theorem 4.1 gives a proof of Theorem 1.7 after a detailed case analysis. This together with the proof of the bounds (4.5)–(4.10) on S_1, S_2, S_3, and S_4 are available in the full version of our paper [5].

5 Lower bound for complete graphs

In this section, we provide a counterpart to Theorem 1.4—a logarithmic lower bound on the expected number of faces Theorem 1.5.

Theorem 1.5. *For all positive integers n, we have*

$$
\mathbb{E}[F(n)] > \tfrac{1}{2}\ln(n) - 2.
$$

Proof. We partition the set of possible (oriented) faces according to their length and we only count those that are easy to count: Let F'_k be the number of (oriented) faces having k distinct vertices and k edges on their boundary. There are $\frac{1}{k}n(n-1)\cdots(n-k+1)$ possibilities for such a face. Each of them becomes a face of a random embedding with probability $(n-2)^{-k}$. Together, we get (using Bernoulli's inequality):

$$\mathbb{E}[F'_k] = \frac{1}{k}\prod_{i=0}^{k-1}\frac{n-i}{n-2} \geq \frac{1}{k}\prod_{i=0}^{k-1}\Big(1-\frac{i}{n}\Big) \geq \frac{1}{k}\Big(1-\sum_{i=0}^{k-1}\frac{i}{n}\Big) \geq \frac{1}{k}\Big(1-\frac{\binom{k}{2}}{n}\Big) = \frac{1}{k}-\frac{k-1}{2n}.$$

Let $m \coloneqq \lfloor\sqrt{2n}\rfloor$. Then $F \geq F'_3 + F'_4 + \cdots + F'_m$, and

$$\begin{aligned}\mathbb{E}[F] &\geq \sum_{k=3}^{m}\mathbb{E}[F'_k] \geq \sum_{k=3}^{m}\left(\tfrac{1}{k}-\tfrac{k-1}{2n}\right) = H_m - \tfrac{3}{2} - \tfrac{1}{2n}(2+3+\cdots+(m-1)) \geq H_m - 2\\ &= H_{\lfloor\sqrt{2n}\rfloor} - 2 \geq \ln(\sqrt{2n}) + \Big(\ln(\lfloor\sqrt{2n}\rfloor) - \ln(\sqrt{2n})\Big) - 2 + \gamma\\ &\geq \frac{1}{2}\ln(n) + \frac{1}{2}\ln(2) + \ln(1/2) - 2 + \gamma > \frac{1}{2}\ln(n) - 2\end{aligned}$$

We have used estimate $H_m \geq \ln(m) + \gamma$ (implied by Theorem 1.11) and $\lfloor\sqrt{2n}\rfloor/\sqrt{2n} \geq 1/2$. ∎

6 Open Problems

We showed that almost all dense graphs have a polylogarithmic average number of faces. Then, in [5, Section 8] we showed that random sparse graphs have a logarithmic average number of faces. By Markov's inequality we conclude that almost all sparse graphs have a logarithmic average number of faces. This leads us to the conjecture that this property holds for almost all graphs, without any density condition on the edges.

Conjecture 6.1. *For any $p(n) : \mathbb{N} \to [0,1]$, almost all graphs in $G(n,p)$ satisfy $\mathbb{E}[F] = O(\ln(n))$.*

This conjecture would follow from the stronger statement of Conjecture 1.2. Conjecture 1.2 can also be stated in terms of the closely related model of random graphs with n vertices and M edges.

Conjecture 6.2. *The expected number of faces in a random embedding of a random graph $G \in G(n,M)$ is*

$$(1+o(1))\ln(M).$$

A main result of the paper was that the complete graph does have a logarithmic number of expected faces. A large family of examples of graphs on n vertices with $\mathbb{E}[F] = \Theta(n)$ are given in [4]. However all of these examples have maximum degree $O(1)$ with respect to the number of vertices. We were unable to find any examples of dense graphs with such a large number of average faces, which leads us to the next conjecture.

Conjecture 6.3. *Let G be a graph on n vertices with minimum vertex degree $\Omega(n)$. Then G satisfies $\mathbb{E}[F] = O(\ln(n))$.*

Theorem 1.7 confirms this conjecture for the complete graph. The multiplicative constant in our bound is not optimal, we restate the conjecture given in the introduction which suggests a possible optimal constant.

Conjecture 6.4. ([32, page 289]) *The expected number of faces in a random embedding of the complete graph K_n is $2\ln n + O(1)$.*

Another natural line of enquiry would be to extend these results to non-orientable surfaces. One natural way to define a random embedding of a graph on a non-orientable surface is to randomly choose a rotation system, and randomly choose a signature for all the edges in the graph, with probability $1/2$ of being either sign. From data, we expect a similar result to hold for non-orientable random embeddings of K_n under this definition.

Conjecture 6.5. *The expected number of faces in a non-orientable random embedding of the complete graph K_n is $\ln(n) + O(1)$.*

We think that in general, a similar property should hold for random embeddings of all graphs.

Conjecture 6.6. *Let F^- be the random variable for the average number of faces in a non-orientable random embedding of some graph G. Then $\mathbb{E}[F^-] \leq \mathbb{E}[F]$.*

It is an easy exercise to check this conjecture's validity on some toy models. In particular, the chain of triangles joined by cut edges considered in [6] satisfies this property. Also, an analysis of Random Process A gives the upper bound of $\mathbb{E}[F^-] \leq \frac{1}{2}\mathbb{E}[F] + 1$ for the dipole, which is the graph with 2 vertices joined by m edges. Computer data ran on some more general graphs gives evidence for some small values of n.

Lastly, it would be of interest to understand higher moments of F. This is widely open even for a complete graph. In this paper, we only obtain an upper bound (with respect to k) for the second moment of the number of potential faces on $n-k$ vertices in K_n.

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