Pre-service teachers' understanding of sine and cosine functions and their inverses based on the unit circle trigonometry

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We applied Action-Process-Object-Schema (APOS) theory to study pre-service secondary mathematics teachers' understanding of the concepts of sine and cosine functions and their inverses based on the unit circle approach. We used a model (genetic decomposition) of mental constructions that students may do to understand these notions and designed research-based didactical materials and implemented them in two countries, Czech Republic and Iran. Eighteen pre-service teachers (nine from each country), who were studying a bachelor's degree of mathematics, participated in our research. The study involves three phases: initial interview, instructional intervention, and exit interview, which were separately carried out in each country. We discussed the teachers' understanding in both initial and exit interviews.

Keywords: Teacher's preparation, Trigonometric, Radian, Unit circle, APOS.

INTRODUCTION

Trigonometric functions and their inverses play a crucial role in understanding many aspects of mathematics, including calculus. Literature review shows, however, that students and teachers have problems in understanding and reasoning about trigonometric functions (Martínez-Planell & Cruz Delgado, 2016; Weber et al., 2020). This suggests that difficulties regarding trigonometric functions among future teachers may impact student understanding in the long term. The teaching and learning of trigonometric functions is not a straightforward process and can be influenced by a number of variables. It is often students' first encounter with a function that does not provide an explicit rule to compute its output. To interpret the values of trigonometric functions for real inputs, students would need to imagine the process of obtaining these values in the unit circle (Martínez-Planell & Cruz Delgado, 2016). Failing to understand this process limits students' knowledge of trigonometric functions and their inverses. To address the difficulties in learning trigonometry, some researchers have focused on angles and the relationship between radian and degree measures in the unit circle (Tallman & Frank, 2020). These studies show that students did not regard the radian as a unit of angle measure and their conception of angles was based on degree measure; therefore, students were unable to interpret the output of trigonometric functions in situations where inputs were given as real numbers. Hence, in our research we focus our attention on radian measure and helping students construct a mental image of sine, cosine and their inverse functions using the unit circle.

LITERATURE REVIEW

Some research has investigated basic constructions related to learning trigonometric functions. Brown (2005) analysed teacher's teaching and students' work regarding radian measure and the unit circle. She observed that understanding the "unit" in unit circle is a fundamental idea that causes student difficulty and is underestimated by teachers during teaching. The students of her study used the phrase 'unit circle' in such a way that was not necessarily accompanied by an appreciation that the radius of the rotation was 1. Bagni (1997) studied students' understanding of trigonometric equations. He reported that 80% of the 67 students in his research could provide a complete or partial solution to easy trigonometric equations (e.g., $\cos x = 1/2$ or $\sin x = -1/2$), by remembering and mentally reversing a memorized table of the values of trigonometric functions. The results also revealed that more than half of the students produced incorrect answers or no answer to questions such as find all real values x in $\sin x = 1/3$ (Bagni, 1997). In this regard, Weber (2005) conjectured that students need to be able to imagine the process of constructing trigonometric functions (from angle to circle to value) which gives rise to the unit circle definition of these functions in order to be able to understand the sine and cosine functions and go beyond the repetition of memorized procedures and facts. Weber also suggested that students should explicitly and physically construct geometric objects to help them deal with trigonometric situations and their inverses. Although inverse trigonometric functions play an important role in many secondary and university mathematics curricula, research on students' and teachers' conceptions of inverse trigonometric functions is limited. In one of them, Weber et al. (2020) investigated 14 pre-service and in-service teachers' understanding of the inverse sine function. They reported that almost all the teachers participating in their study were not able to explain how by restricting the domain of the sine function, an inverse function is possible.

THEORETICAL FRAMEWORK

The study used APOS theory (Arnon et al., 2014) as the theoretical framework. In APOS theory, an Action is a mathematical transformation that the student perceives as external. An Action may be the rigid application of an explicitly available algorithm or of a memorized formula or procedure. When an Action is repeated, and the student reflects on the Action or on a chain of Actions, it might be interiorized into a Process. A Process is perceived as internal and allows the student to omit steps, anticipate results, and thus generate dynamical imagery of the Process. Different Processes may be coordinated or reversed to form new Processes. When a student is able to think of a Process as a whole and is able to do Actions or imagine doing Actions on that whole, then one says that the Process has been encapsulated into an Object. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas having to do with a particular mathematical notion. Research in APOS typically starts by proposing a hypothetical model (a conjecture) in terms of the structures and mechanisms of the theory of how a generic student may construct a specific mathematical notion. This model is called a genetic decomposition (GD). The following is the portion of the GD for a unit circle based introduction to the sine, cosine, and their inverses designed by Martínez-Planell and Cruz Delgado (2016) and that was again used for this study in order to design and implement activities to help students do the proposed constructions.

Process #1: $t \to P(t)$ process. Construction of the sine and cosine functions may start with the action of taking a given real number t and locating, as a geometric representation, the terminal point $P(t)$ of an arc along the unit circle that starts at the point $(1,0)$, has length $|t|$ radii and is traversed either counter clockwise when $t \geq 0$ or clockwise in the case that $t < 0$. As students repeat and reflect on this action they may be able to imagine taking any given real number t and assigning to it a point $P(t)$ on the unit circle without having to do so explicitly. In this case they can be said to have interiorized the action into a process, denoted $t \to P(t)$.

Process #2: Circ process. In another construction, given a point $P(t)$, represented geometrically or as an ordered pair, the student can perform the action of finding the other three corresponding points $P(-t)$, $P(t+\pi)$, and $P(\pi - t)$ on the unit circle, geometrically or as ordered pairs. These actions are interiorized into a process that enables students to locate on the same circle the geometric representation of any point of the form $P(t + 2k\pi)$, $P(-t + 2k\pi)$, $P(t + \pi + 2k\pi)$, $P(\pi - t + 2k\pi)$ when they know a geometric representation for the point $P(t)$.

Process #3: projection process. Now a process conception of the sine and cosine functions may be constructed by coordinating the $t \to P(t)$ process with a corresponding projection process. Projecting onto the y axis [x axis] defines the sine [cosine] function. These actions of projection may be interiorized into processes of "projection". The processes of locating a corresponding point $P(t)$ and then projecting onto a corresponding axis (as described above) may be coordinated into processes which we will refer to as the definition of the sine and cosine functions.

Process #4: reversal of the projection. To reverse the projection of the sine function, start with a number k in the interval $[-1,1]$ and perform the action of representing a point on the y axis that has k as its ordinate. The next action is to locate on a physical or geometric representation of the unit circle all the points that are projected horizontally onto $(0, k)$. To reverse the projection of the cosine function, perform the analogous action, namely represent a point on the x axis having abscissa k, and then identify all points on the unit circle that project vertically onto $(k, 0)$. Repetition and reflection on these actions may be interiorized into a process of projection reversal.

Process #5: reversal of the $t \to P(t)$ process. To reverse the $t \to P(t)$ process, the student starts with one or two points $P(t)$ resulting from a projection reversal and finds a value of t determining one of the points. At this stage, finding an approximation of a real number t that determines a point $P(t)$ with a specific x or y coordinate may be done physically with a manipulative like a piece of ribbon.

Process #6: reversal of the definition. After a student reverses a projection and obtains the points that correspond on the unit circle, the student may coordinate the reversal of the $t \to P(t)$ process (to obtain one value of t) with the Circ process to obtain all values of t that determine the points he/she found on the unit circle. The reversal of the $t \to P(t)$ process followed by the coordination with Circ results in a process that allows the student to recognize that the sine and cosine functions are not one to one. The chain of actions that starts with a number, represents it as an x or y coordinate on the corresponding axis, goes on to identify the point or points on the unit circle having that number as an x, or y coordinate, and then identifies all the real numbers corresponding to the point or points on the unit circle, may be interiorized into a process that we will call reversal of the definition. This process starts with a coordinate and produces the collection of all real numbers corresponding to the points on the unit circle (one, two, or none) having that coordinate. By its nature this process does not define a function.

Process #7: Range process. To construct a process conception of Range, students could interiorize actions that explore ways of restricting the domain of the sine and cosine functions to an interval so that the resulting function is one to one and the restricted domain is as large as possible. These actions should include both, the unit circle representation and the graphs of these functions. Students that interiorize these actions into a process would recognize the need to restrict the domains of sine and cosine as well as the convenience of restricting these domains as they normally are. Students not able to argue for the need of a restriction and reasonableness of the usual restrictions of the domains of sine and cosine will be constrained to having an action conception of Range as a memorized fact.

METHOD

We conducted the study separately in two countries: Czech Republic and Iran. Nine pre-service secondary teachers in their second year of an undergraduate mathematics field from a teacher training department in Czechia, and nine pre-service secondary teachers in their second year of an undergraduate mathematics field from a teacher training department in Iran voluntarily participated in this study. We did the study in these two countries because we had the opportunity and were interested in seeing how students from different cultures and backgrounds react on the interview questions and the set of class activities. However, we did not aim to do a quantitative comparison of students' understanding between the two countries because the amount of data was not enough to make any meaningful comparison. Furthermore, in order to make such a comparison, many other factors and variables would have to be examined, which were out of the scope of this study. The students in both countries learned the trigonometric functions and their inverses in several courses in secondary school and university and applied these concepts to study other advanced mathematical notions (e.g., limit, derivative, and integral). Our study consisted of an initial interview, three sessions of instruction, and an exit interview. After the initial interview, we started the instruction phase in both groups (one in Czechia and the other in Iran), and one week after the instruction we conducted the exit interview. For the instruction, we designed a set of activities to help the pre-service teachers interiorize the geometric processes conjectured in the GD, thus extending the study of Martínez-Planell and Cruz Delgado (2016). The activities were implemented in three class sessions (each 90 minutes) by two experienced instructors (one in Czechia and the other in Iran). During the lessons, the participants frequently used a piece of ribbon and a circle to solve the activities. In the first class, the pre-service teachers used a ribbon, to prepare a measuring tape in units of circle radii. They used the measuring tape to measure the number of radii in a half circle and found the relationship between this number and the value of π, and also the number of radii in the circumference of the circle and compare it with the well-known formula $S = 2\pi r$. Regarding the $t \to P(t)$ process, students used a ribbon to locate points $P(t)$ on the circle determined by t (radii). Then, for a point on a circle they determined three other points on the circle, reflecting through each axis and through the origin (Circ process). In the second class, they represented the projection of $P(t)$ on the y axis [x axis] in units of radii to find the value of sine and cosine of the real number t (projection process). They extended the definition of sine and cosine directly to the acute angles of a right triangle where they thought of hypotenuse as the radius of a circle. In some activities they used the sine and cosine functions on the unit circle to draw the graph of these functions in the Cartesian coordinate system. In the third class, they constructed the inverse sine and cosine functions. In this regard, they started with a number between -1 and 1 as input and produced points in a unit circle as output (from number to point, reversal of the projection). Then they started with a point on a unit circle as input and using a ribbon as a measuring tape produced an angle measure as output (point to angle, reversal of the $t \to P(t)$ process). The combination of the two previous steps, starting with a number and ending with an angle, is used to reverse sine and cosine. For this, they also did some activities to find the conditions for uniqueness when reversing sine and cosine in the unit circle and Cartesian coordinate system (Range process).

The initial and exit interviews were audio recorded. The interviews in both groups (i.e., Czechia and Iran) were done by one of the authors of this paper who separately gathered the data in each country. The interviews were audio and video recorded, transcribed, translated to English, analysed individually, and discussed as a group. Differences in opinion were negotiated. The individual and group analysis concentrated on trying to ascertain the mental structures (Actions, Processes, Objects) evidenced by students in regards to the GD. The initial interview questions were: **1)** Show angles measuring $20°$, $\pi/2$ radians, and 4 radians in the unit circle. **2)** $\boldsymbol{a}$**.** Find the value of $\sin 30$. $\boldsymbol{b}$**.** Find the value of $\sin 30°$. **3)** Using Figure 1(a), determine a formula between r, θ, and S and justify it (r is the length of the radius, θ is in radians, and S is the length of the given arc) **4) a.** Show the answers of $\sin x = 1/3$ ($x \in [0,2\pi]$) on the unit circle. **b.** Why are $\sin(\pi/2)$ and $\cos(\pi/2)$ equal to 1 and 0, respectively? **5)** Find approximately $\cos(2.5)$ using the following circle (Figure 2(a)). **6)** Find $sin^{-1}(-1)$ and justify your answer.

The exit interview questions were: **1)** Given that the angle θ in Figure 1(b) measures 0.6 radians, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2.2 cm, 4.2 cm, and 6.2 cm. **2)** Use the unit circle (in Figure 2(a) and a piece of ribbon to approximate the value of $\sin(1.2)$ the best you can. **3)** Put in order from lowest to highest: cos(1), cos(1/10), cos(3), cos(−4), cos(4), cos(6). **4)** Find all the solutions of $\cos(t) = -3/5$. **5)** Given that $\cos(22\pi/7) = -0.90096887$. Find $cos^{-1}(-0.90096887)$. **6)** Use the unit circle to show: $sin^2 x + cos^2 x = 1$

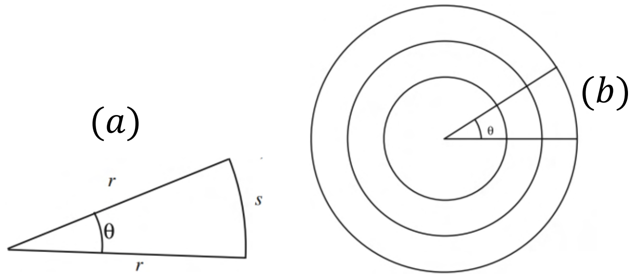

Figure 1: (a) related to Q3 of the initial interview, (b) related to Q1 of the exit interview.

RESULTS

We start by considering the number of the pre-service teachers (in both countries) constructing the different Processes of the GD in the initial and exit interviews (Table 1). The data in Table 1 shows that the teachers did not show evidence of most of the mental constructions of the GD in the initial interview. However, their answers to the exit interview improved compared to those of the initial interview.

GD / #students in:	Process #1	Process #2	Process #3	Process #4	Process #5	Process #6	Process #7
initial interview	3/18	4/18	5/18	2/18	2/18	2/18	2/18
exit interview	15/18	13/18	16/18	13/18	12/18	11/18	11/18

Table 1: Number of the teachers constructing specific parts of the GD

Initial interview results. Most of the pre-service teachers had significant challenges with the mathematical tasks involved in the initial interview. Indeed, none of them can be classified as having a process conception of sine and cosine. Only 2 of the 18 interviewed teachers are classified as being in transition level between the action and process conceptions, and another 16 teachers seemed to have been limited to an action conception. Here we consider some typical answers of the teachers to questions 1, 2, 3, 5, and 6. All the teachers from both countries showed the angles $20°$ and $\frac{\pi}{2}$ radians correctly. However, 15 teachers (eight from Iran and seven from Czechia) were not able to draw the angle of 4 radians. The answers of Jan and Keyvan were typical responses of these pre-service teachers:

Jan: [from Czechia] I think 4 radians is another notation for 4π, so it's on the positive side of x axis.

Keyvan: [from Iran] I don't know in which quadrant 4 radians is, actually I always have problem with radian, I don't know what it means, it's more convenient for me to use degree for angles, for example I directly and without any computation know the angle four degrees is in the first quadrant but I can't immediately realize where the angle of four radians is, but I need to do some computations, I know that π is approximately 3.14, and 4 is bigger than π so the angle 4 radians is probably in the third quadrant.

Jan's answer is consistent with the findings of Bagni (1997) where students in her study considered π as the unit for the radian measure and considered 1 radian equalled to $180°$. Also, like Keyvan, some of the teachers in both countries did not consider π as an angle measure related to the arc length of a circle. The data indicated that the pre-service teachers were more comfortable with degree measure than radian measure. Related to this issue, almost all the teachers (except one) did not realize that 30 in $\sin 30$ (question 2 part a) is expressed in radian and considered ½ as the correct answer for $\sin 30$. Although when facing with $\sin 30°$ in part b they corrected their answer and put $\sin 30° = 1/2$, they still were unable to discuss how to get the approximation of $\sin 30$ using the unit circle. The pre-service teachers still need to construct the $t \rightarrow P(t)$ process to measure angles in radians and coordinate it with the projection process to find the values of sine and cosine of real numbers. In question 3 we asked the teachers to determine a formula relating r, θ, and S and to justify it. Eleven pre-service teachers did not find the correct formula. The other 7 teachers (four from Iran and three from Czechia) found the formula using proportions and not as stemming from understanding radian measures as a multiplicative relationship between arc length and radius length. Monika's answer was typical.

Monika: [from Czechia] We know that the circumference of a circle with radius r is $2\pi r$ and the angle is 2π, umm so when the angle is θ we need to find S, So we just need to solve the proportion $\frac{2\pi}{\theta} = \frac{2\pi r}{S}$ for S. It will be $S = r\theta$.

Interviewer: Can you explain more about this formula in terms of the unit circle?

Monika: Oh, for me it's just an algebraic formula between variables r, S, and θ.

Monika's explanations showed that she did not think of the radian measure of the central angle θ as the number of radii in the corresponding arc-length. This is consistent with not constructing the $t \rightarrow P(t)$ process which helps students think of the measure of a central angle in a circle of any radius as the number of radii in the corresponding arc-length. In question 5 (approximating $\cos(2.5)$), none of the teachers (in both countries) gave correct explanations. Here we consider two of such responses.

Hamid: [from Iran] The angle 2.5 degrees is very close to 0, so $\cos 2.5$ is approximately close to $\cos 0$ which is 1.

David: [from Czechia] We learned to find the values of trigonometric functions for angles such as $\pi/6$ and $\pi/4$, umm I don't know how to find the cosine of 2.5, in such cases I use calculator.

Interviewer: Can you explain based on the unit circle approach why $\cos \pi/4$ is $\sqrt{2}/2$?

David: No, I don't, even for these convenient angles like $\pi/6$, $\pi/4$, or $\pi/3$ I just memorized the table of the values of trigonometric functions.

Most of the pre-service teachers, like David, just memorized the trigonometry table and were not able to justify the value of sine and cosine of neither convenient angles (e.g., $\pi/6$ and $\pi/4$) nor real numbers that are not integer multiples of $\pi/6$ and $\pi/4$ (e.g., 2.5 in $\cos(2.5)$). In question 6 (finding the value of $sin^{-1}(-1)$), three teachers (two from Czechia and one from Iran) found $-\pi/2$ as the correct answer using a memorized fact, showing an action conception of Range. The other pre-service teachers showed some problems with this question. We consider the answers of Samira and Kristyna as typical answers of such students.

Samira: [from Iran] I need to find an angle where its sine is -1, umm so it can be $-\pi/2$ and $3\pi/2$, or generally $-\pi/2 + 2k\pi$.

Kristyna: [from Czechia] Let's put $sin^{-1}(-1)$ equal to θ, umm by taking the sine from both sides we have $\sin(sin^{-1}(-1))$ equal to $\sin\theta$, umm I know $f(f^{-1}(x))$ is x, so $\sin(sin^{-1}(-1))$ is -1. I changed the question to the easier question $\sin\theta = -1$, we can find many angles as the answer, for example $3\pi/2$.

Both Samira and Kristyna had not constructed the Range process. The same difficulty was observed in most pre-service teachers.

Exit interview results. One week after the instruction in each group an exit interview was taken from the pre-service teachers in each country. In short, we started with 18 pre-service teachers who showed several difficulties with the construction of sine and cosine and their inverses based on the unit circle approach as evidenced by the results of the initial interview, and after an instructional intervention most of them showed an improvement in their understanding. Indeed, of the 18 interviewed teachers, we classified 10 as having reached at least the process conception of sine, cosine, and their inverses; 5 a transition level between the action and process conceptions; and 3 remained at the action conception. Here we show some typical answers of the teachers to questions 1, 3, and 5 in the exit interview. Fourteen teachers (eight from Czechia and six from Iran) determined the length of each arc cut off by the angle in question 1. We consider the case of Monika:

Monika: The angle is 0.6 radians, it means in the corresponding arc of each circle there are 0.6 radii of that circle, so for each circle 0.6 times the radius length of that circle is equal to the corresponding arc length, 0.6 time 2.2, and 0.6 times 4.2, and also 0.6 times 6.2 cm will be the length of each arc cut off by the angle.

Monika's response gives evidence of understanding the ideas behind the usual formula for arc-length in terms of radian measure and radius length (i.e., $S = r\theta$) in a way which is not dependent on the manipulation of a symbolic expression (her response in the initial interview), but rather the ideas may surface as natural relations that do not require memorization. This gives evidence of constructing the $t \rightarrow P(t)$ process. Regarding question 3, thirteen pre-service teachers (seven from Czechia and six from

Iran) ordered the values of the cosine function correctly. We consider the case of Hamid (Figure 2(a)) who was not able to find cos(2.5) in the initial interview.

Hamid: I need ribbon to find the location of these radians as points on the unit circle, then I project each point onto the x axis to find the cosine of each radian.

Like Hamid, a high proportion of the teachers under this experimental instruction gave evidence of the $t \rightarrow P(t)$, Circ, and projection processes and their coordination to estimate the values of the trigonometric functions of non-standard angles, articulate the process of finding the value of the sine and cosine of an angle, as well as to derive and explain properties of the trigonometric functions. In question 5, students had $\cos(22\pi/7) = -0.90096887$ and we asked them to find $cos^{-1}(-0.90096887)$. Consider the answer of Samira (Figure 2(b)) who had problems with the inverse of trigonometric functions in the initial interview.

Samira: We can write $22\pi/7$ as $3\pi + \pi/7$, so the angle is in the third quadrant, now I project to the x axis, the range of cos^{-1} is from 0 to π to be a one-to-one function, so the answer of $cos^{-1}(-0.90096887\ )$ can't be $22\pi/7$, I need to reflect $22\pi/7$ respect to the x-axis, the final answer will be $\pi - \pi/7$ in the second quadrant. I can see both $\cos 22\pi/7$ and $\cos(\pi - \pi/7)$ are equal to -0.90096887 but the cosine inverse of -0.90096887 is only $\pi - \pi/7$.

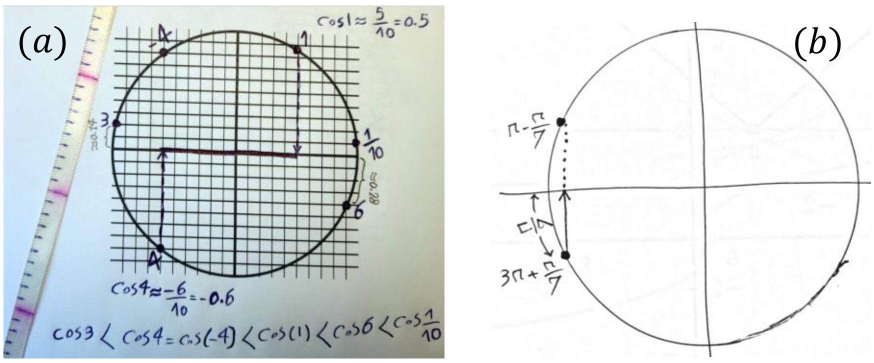

Figure 2: (a) Hamid's response to question 3, (b) Samira's response to question 5.

Overall, the pre-service teachers gave evidence consistent with the Range process when they recognized that they were looking for a value in the interval from 0 to π.

DISCUSSION AND CONCLUSION

This study was designed to examine aspects of the pre-service teachers' understanding of sine and cosine functions and their inverse based on the unit circle approach and the effect of a set of activities, designed to help construct these concepts as proposed in the GD. We found that in the initial interview, the pre-service teachers were not yet able to draw angles given in radian measure, approximate values of sine and cosine for non-integer multiples of $\pi/6$ and $\pi/4$, and apply inverse trigonometric functions. Based on the initial interview, the teachers did not express sufficient knowledge of trigonometric functions to teach the topic. However, the research-based activities, designed to help students do the constructions proposed in the GD, succeeded in having most participants overcome the difficulties observed in the initial interview and show improvement as documented in the exit interview. This is an important contribution of our research. Indeed, this study contributes to a better understanding of how students may construct the notion of sine and cosine functions and their inverses based on the unit circle approach. Our research is in line with the studies carried out by Tallman and Frank (2020) where they studied secondary teachers' knowledge of sine and cosine values and reported their difficulties. However, we add to their contribution by considering teachers' understanding of inverse trigonometric functions, emphasizing the importance of teachers' being able to imagine reverting the definition of sine and cosine (from number, to corresponding axis, to points on the circle, to considering range, to arc-length and angle). Like Weber et al. (2020) we considered the process of inverse trigonometric functions, however, we also investigated the influence of research-based activities on learning radian measure, sine and cosine functions, and the inverse of these functions. Finally, the results suggest that, in general, the participating students in Iran and Czech Republic tended to construct the same type of structures dealing with basic trigonometry. In future work, we will refine the GD to include relating angle measurements in degrees and radians, the relation between trigonometric ratios in unit circle and in right triangle trigonometry, and the basic construction of the sine and cosine functions and their inverses with their graphical representation.

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