Building on slope: results of a second research cycle on the differential calculus of two-variable functions

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This study applies Action-Process-Object-Schema (APOS) theory in a second research cycle investigating how an approach based on the explicit consideration of slope in three dimensions and local linearity contributes to students' learning of the differential calculus of two-variable functions. We compare the problem-solving tendency of students from two sections that were taught differently. We find that students' conceptualizations of slope were different in the two sections and that their understanding of slope was reflected in their problem-solving and the justifications of the relations that they constructed between different basic notions of the differential calculus. Overall, we show that the students who based their constructions on slope and local linearity obtained a deeper understanding of the differential calculus.

Keywords: APOS, functions of two variables, slope, differential calculus, multivariable calculus.

INTRODUCTION

Slope is a basic notion that is commonly studied for the first time in the middle school curriculum, then is revisited in secondary school, in courses such as algebra, trigonometry, and pre-calculus, before being revisited again in the context of the calculus of one-variable functions (Nagle et al., 2019). So, it is reasonable to attempt to base students' understanding of multivariable calculus on the notion of slope. However, students can show difficulty generalizing the notion of slope from two to three dimensions (Martínez-Planell et al., 2015; Moore-Russo et al., 2011). So, attempts to build the differential calculus of two-variable functions based on the notion of slope require the explicit consideration of slopes in three dimensions when teaching the course (McGee & Moore-Russo, 2015). This is done in this study, in which classroom instruction of an "activity section" started with an explicit discussion of slope in three dimensions (3D), then slopes were used to have students construct a notion of vertical change on a plane ($dz = dz_x + dz_y$; see Figure 1) which subsequently served as a basis from which to develop other ideas of the differential calculus of two-variable functions, including the tangent plane, directional derivatives, and the total differential. In the study, we investigate how this approach to multivariable calculus affects students' ways of thinking about slope and, more importantly, how students who used this approach tended to establish more relations between different notions of the differential calculus of two-variable functions when problem-solving justifying the relations in terms of slope, in comparison with students that did not use this approach.

There is an increasing number of studies dedicated to the didactics of multivariable differential calculus (e.g., Borji et al., 2023, 2024; Harel, 2021; Lankeit & Biehler, 2019; McGee & Moore-Russo, 2015; Martínez-Planell et al., 2015, 2017; Tall, 1992; Trigueros et al., 2018; Weber, 2015). But there is still much to learn, particularly as it pertains to how can one best help students to interrelate the different notions of the differential calculus of two-variable functions.

THEORETICAL FRAMEWORK

We use the Action-Process-Object-Schema theory (APOS). For more details, see Arnon et al. (2014). In APOS, an Action is a transformation of a previously constructed mathematical object that the individual perceives as external. An Action may appear as a rigid application of an explicitly available or memorized fact or procedure. When an Action is repeated, and the student reflects on the Action, it might be interiorized into a Process. A Process is perceived as internal. A student with a Process conception will show characteristics like justifying the Process, discussing it in general terms, thinking of it as independent of representation, and generating dynamical imagery of the Process. A Process may be reversed or coordinated with other Processes to form new Processes. When an individual is able to think of a Process as an entity in itself and can apply or imagine applying actions on this entity, then the Process has been encapsulated into an Object. The important thing about an Object is being able to do Actions on a (previously encapsulated) Process. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas that are interrelated in such a way that the individual can determine if it applies to a particular problem situation. Although the complexity of the differential calculus of two-variable functions suggests the use of Schemas to model student understanding, in this report we focus on slope and its role in establishing connections between different component structures of the differential calculus of two-variable functions. We will not need to use Schemas to model this.

Another important idea in APOS is that of a genetic decomposition (GD). This is a model of constructions a student could do in order to understand a particular mathematical notion. The GD is expressed in terms of the structures (Action, Process, Object, Schema) and mechanisms (interiorization, coordination, reversal, encapsulation, de-encapsulation, etc.) of the theory. A GD is not unique and is not meant to be the best way a student may come to understand a notion. It is only a hypothesis that may be improved by research results. After proposing a GD, classroom activities are designed to help students do the proposed constructions. They are class-tested, and data is obtained from students with an instrument based on the GD. The obtained data can suggest improvements to the GD and the activities. The new GD and activities may be tested in further cycles of research.

Reflection is the key ingredient allowing students to go beyond an Action conception. To foment reflection, in APOS one typically uses the ACE pedagogical strategy. This means that the specially designed activities are worked in collaborative groups of three or four students, there are general class discussions, and exercises for the home.

GENETIC DECOMPOSITION (GD)

We adapt some ideas of Tall (1992) to base the development of the differential calculus of two-variable function on the notion of slope and local linearity. The construction is suggested by Figure 1 and is developed in much detail in a GD given by Martínez-Planell et al. (2017), and for the total differential, in Trigueros et al. (2018). We must omit all details for reasons of space. Essentially, we start by the explicit consideration of slope m in three dimensions (3D). Since we treat a surface as locally linear, we start by considering planes and use the slopes in the x and y directions, m_x and m_y, to construct Processes of vertical change on a plane in the x and y directions, $dz_x = m_x dx$ and $dz_y = m_y dy$ respectively, and coordinate them to obtain a Process of total vertical change on a plane $dz = dz_x + dz_y$. From here, the point-slopes equation of a plane follows immediately and if the plane happens to be the tangent plane, we also obtain its equation, were the slopes in the x and y directions are now the partial derivatives. As Figure 1 suggests, the notions of total differential and directional derivative can also be obtained based on this idea.

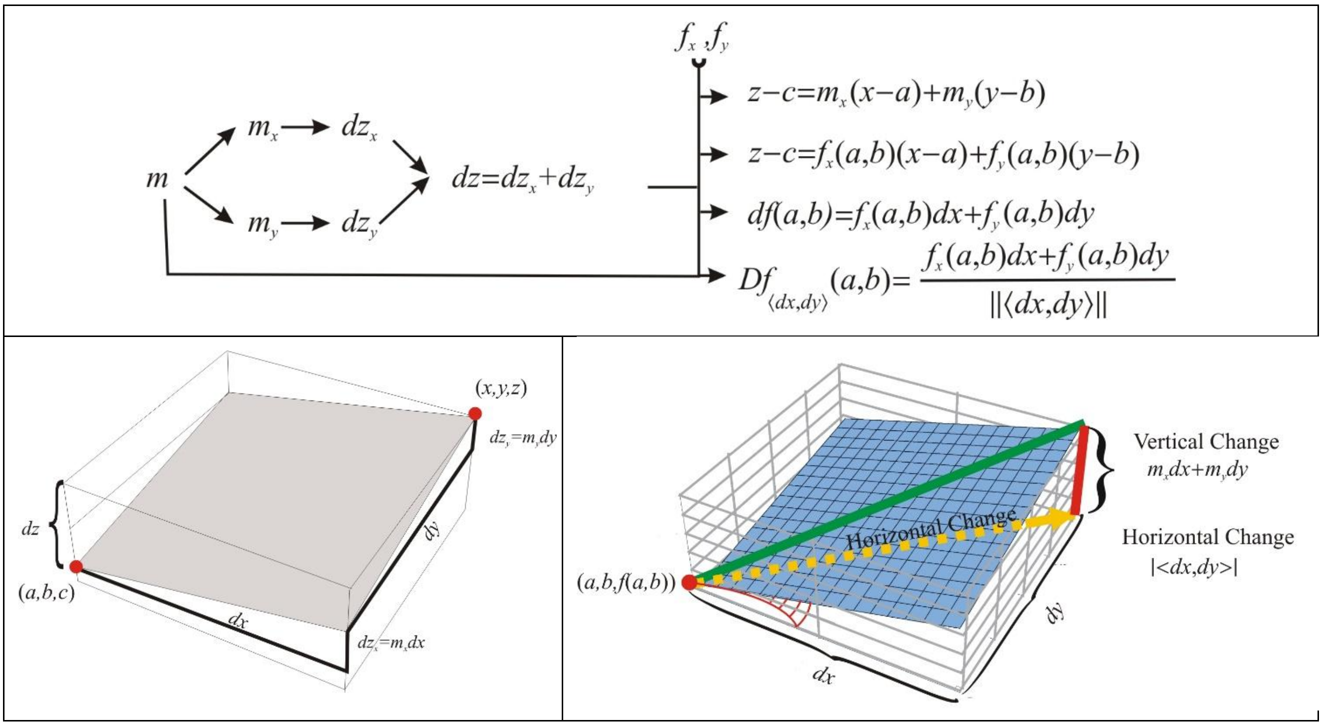

Figure 1: Diagram and figures suggesting the main constructions of the Differential Calculus (*m* is slope).

FIRST RESEARCH CYCLE AND METHODOLOGY

This study is a result of a second research cycle investigating students' understanding of the differential calculus of two-variable functions. The first research cycle and the resulting GD are described in Martínez-Planell et al. (2015, 2017) and in Trigueros et al. (2018). The results of the first cycle showed that students seemed to have an Action conception of the main ideas of the differential calculus; many students did not construct slope in 3D and they seemed constrained to the rigid application of memorized formulas which they could not justify geometrically. The problem is that visualization is not possible when working at the Action level. Only one student (out of 26) constructed a Process of directional derivative, and none constructed a Process of total differential. The data suggested that the notions of tangent plane, total differential, and function remained isolated in the minds of most students.

After that first cycle, the GD was revised and the activity sets were redesigned in order to better help students do the constructions proposed in the GD and use slope and the tangent plane to interrelate partial and directional derivatives, the point-slopes equation of a plane, and the total differential in different representations as described before. The GD was now more detailed, thus introducing many changes in the activities. The activities for the differential calculus and other areas can be downloaded in the link https://www.researchgate.net/publication/373990320_Activities_for_Multivariable_Calculus.

In this second cycle, we compared students' inferred mental constructions in an activity section that used the newly improved activity sets and the ACE pedagogical strategy, and a regular section that followed very closely the textbook (Stewart, 2012), used problems from the textbook, and was lecture-based. Having a regular section allowed us to recreate conditions similar to those of the first research cycle so that the types of constructions students in the regular section make can serve as a baseline with which to compare the constructions of the students in the activity section. It also enables us to verify that the results of the first research cycle are reproduced with those of the regular section. We underscore that this is a qualitative rather than a quantitative study in which we look for the general tendency of students of the regular and activity sections when constructing different structures (Actions, Processes, Objects) in their problem-solving.

Each professor chose 11 students so that three were over-average, five average, and three under-average according to the professor's criteria. This was done in order to be able to observe as many different types of constructions as possible. The students were comparable in the sense that they took the previous single-variable calculus course with the same professor (of the regular section), and it was verified that they had comparable grades in that course. Both professors had ten years of experience teaching the course. Semi-structured interviews took place after the semester was over; each interview had two parts, which were held on separate days, and each part lasted approximately one hour. The interviews were audio and video recorded, transcribed, and translated into English. The data analysis compared the structures (Action, Process, Object) that students gave evidence of having constructed with those proposed in the GD, and also took note of unconjectured constructions. The analysis was done individually by the researchers and then discussed as a group until a consensus was reached.

The interview instrument had a total of 20 questions in its two parts. For the purpose of this article and for lack of space, we only show the four questions below.

2. a. In the plane given below, find the slope of the line in bold [Figure 2 left].

6. The plane in the figure below [Figure 2 right] is tangent to the graph of a differentiable function $z = f(x, y)$ at the given point.
   a. What can you say about the change in the value of the function if x increases 0.02 units and y decreases 0.02 units?
   b. Find the differential of f at the point (1, 2), $df(1,2)$. If it is not possible, explain why.
   d. Use the graph of the given tangent plane to find $D_{\langle 1,1\rangle} f(1,2)$.

RESULTS

As could be expected, classroom activities that emphasize a geometric interpretation of slope and the notions of differential calculus had an effect on the students' conceptualizations of slope (Moore-Russo et al., 2011; Nagle et al., 2019) with most students in the activity section (10 of 11) giving evidence of a geometric ratio conceptualization of slope while a majority of students in the regular section (7 of 11) showed an arithmetic ratio conceptualization. Of course, as argued in Nagle et al. (2019), students with a Process conception of slope can exhibit either conceptualization as needed in a problem situation. The following two examples show the difference between the geometric and arithmetic ratio conceptualizations of slope. Student A1 is from the activity section and R2 is from the regular section.

A1: The slope of this line will be this vertical change which is 5 minus 2 and it's 3 umm over this horizontal change which is umm from 1 to 2 so the horizontal change is 1. The slope is $\frac{3}{1} = 3$.

Student A1 shows a geometric ratio conceptualization of slope in 3D while, in the following example R2 shows an arithmetic ratio conceptualization.

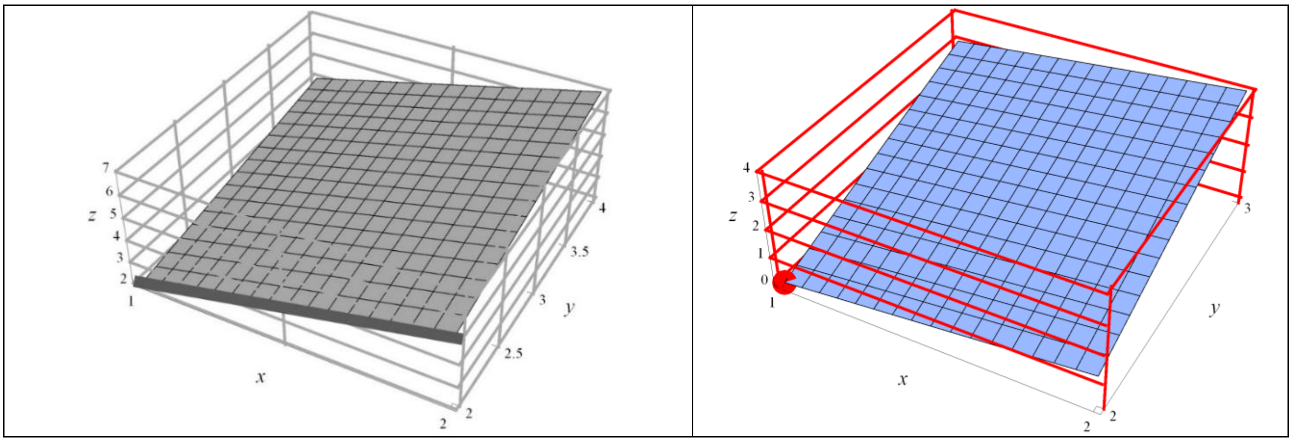

Figure 2: Figures for questions 2a and 6a, 6b, 6d, respectively.

R2: It's a line in $3D$ umm I don't know how to compute the slope of a line in $3D$ because we have three variables x, y, and z in $3D$… The y coordinate is fixed at 2 in both points … So I ignore y in my computation, and I use the formula $\frac{z_2-z_1}{x_2-x_1}$ …

Later, when asked about the slope in the *y* direction:

R2: Okay, I have a line in $3D$ which is in the y direction … I see the x coordinate is fixed as 2 in two points so I ignore it in my computations, and maybe the formula for slope in this case can be $\frac{z_2-z_1}{y_2-y_1}$ … in part 2a [the x direction] the y coordinate was fixed, and in part 2c [the y direction] the x coordinate was fixed, but I am thinking how can we find the slope of a line in $3D$ if all the three coordinates x, y, and z change from first point to the second point. I have no idea for finding the slope of such points because in these cases the formulas $\frac{z_2-z_1}{x_2-x_1}$ and $\frac{z_2-z_1}{y_2-y_1}$ don't work.

Overall, R2 seems to be more dependent on formulas. The statement of R2, "I have no idea for finding the slope of such points because in these cases the formulas $\frac{z_2-z_1}{x_2-x_1}$ and $\frac{z_2-z_1}{y_2-y_1}$ don't work", is an example of "the two change problem," observed early by Yeruschalmy (1997) and explored and named as such by Weber (2015). It anticipates some of the challenges students face if learning directional derivatives by using an entirely algebraic perspective.

Some students in the Regular section showed a geometric ratio conceptualization of slope, like R1.

R1: The line in bold goes 3 units up and moves 1 unit in the *x* axis so its slope is $\frac{3}{1}$.

Overall, as shown in Table 1, all 11 students in the activity section showed their understanding of slopes in 3D by computing the slopes in the *x* and *y* directions, while five of the 11 students in the regular section were also able to compute slopes in 3D, something which, as observed by Moore-Russo et al. (2011) and as seen in the first research cycle (Martínez-Planell et al., 2015), is not generalized on their own by some students. Table 1 also shows relations established between the notions of tangent plane and function (TP-F; problem 6a), total differential (TP-TD; problem 6b), and directional derivative (TP-DD; problem 6d) when problem-solving.

Students showing construction	Slope in 3D	TP-F	TP-TD	TP-DD
Activity section	11	8	8	9
Regular section	5	2	1	3

Table 1: TP=tangent plane, F=function, TD=total differential, DD=directional deriv.

We now consider some examples of these relations. In question 6a, students are given a graphical representation of the tangent plane in order to approximate a change in function values. Student A1 uses slope in his argument, interrelating different notions of differential calculus. He interprets the tangent plane in terms of the total differential, showing awareness that the total differential at the point will give vertical change on the tangent plane as a function of the horizontal change (dx, dy). Further, he relates the partial derivatives to the slopes of the plane in the *x* and *y* directions and ends up by relating total differential to function in order to produce the requested approximation.

A1: I know $df = f_x dx + f_y dy$. Here we have $dx = 0.02$ and $dy = -0.02$. I have to find the values of f_x and f_y at the point (1,2,0). Since it is a tangent line to the function *f* at the point (1,2,0) so f_x is m_x and f_y is m_y. Based on the figure m_x is 1 over umm 2 minus 1 which is 1 so it will be 1, so m_x is 1, and m_y is 3 units to the up over 3 minus 2 which is 1 umm it will be 3 over 1 which is 3, so m_y is 3. The change in the value of the function is 0.02 times 1 plus -0.02 times 3 and umm the answer is -0.04.

Like A1, eight of the 11 students in the activity section also used slope to relate tangent plane and function in question 6a, with some also relating these notions with the total differential. On the other hand, only two of the 11 students in the regular section could do the problem, and none used slopes. Consider R1:

R1: I think this is like Question 2 but here it's tangent plane to the function f. I can find the change in the z coordinate on the plane. Looking at the plane we see when x increases 1 unit umm from 1 to 2 then the z coordinate increases 1 unit to the up umm now if x increases 0.02 units we have the proportion $\frac{1}{0.02} = \frac{1}{\Delta z}$ so it will be $\frac{0.02\times 1}{1}$ which is 0.02. Based on the figure if y increases from 2 to 3 which is 1 unit then the z of the plane increases as 3 units, so for $\Delta y = -0.02$ because y decreases it's negative, so we have the proportion $\frac{1}{-0.02} = \frac{3}{\Delta z}$ and from this we have $\Delta z = \frac{-0.02\times 3}{1}$ which is -0.06. So the final change in z is umm 0.02 minus 0.06 which is -0.04. It's the change of the value of the z of the plane.

Note that R1 relates the tangent plane with the function, showing awareness that one can be used to locally approximate the other. She does this without explicitly recurring to slopes. Instead, she uses proportions.

Question 6b gave students the graph of the tangent plane at a point and asked for the total differential at the point. As shown in the previous problem, A1 had constructed a relation between function, total differential, and tangent plane.

A1: It's $df(1,2)$ equal to 1 times dx plus 3 times dy

Like A1, eight of the 11 students in the activity section could relate tangent plane with total differential. The only student of the regular section to do so was R1.

R1: I know the formula of the differential of f is $df = f_x dx + f_y dy$ and for the point (1,2) it will be $df(1,2) = f_x(1,2)dx + f_y(1,2)dy$. But I don't know how to find $f_x(1,2)$ and $f_y(1,2)$.

Interviewer: Use the figure of the tangent plane.

R1: Since it's the tangent plane to the function f at the point (1,2) umm it seems that I can find the derivatives based on the figure. In Question 2 I found the slope in the x and y direction, now the slope in the x direction is $\frac{1}{1}$ which is 1, and the slope in the y direction is $\frac{3}{1}$ which is 3. If I consider f_x equal to 1 and f_y equal to 3 then the differential will be umm $d_f(1,2) = 1d_x + 3d_y$.

Note that with a hint from the interviewer, R1 was able to find the total differential, perhaps as an Action, a memorized formula, since she did not justify it on her own. In doing so, she showed the need to construct slope as an Object she can flexibly use to relate the graphical representation of tangent plane and partial derivatives. The most common response of students in the regular section, as those in the first research cycle, was similar to that given by R2:

R2: I don't know how to find $df(1,2)$. I don't know what the differential means on the graph umm neither know its formula.

Question 6d gave students the same graph of a tangent plane and this time asked for a directional derivative. Note that the notion of slope is central to A1's argument.

A1: It's the directional derivative. The direction vector is $\langle 1,1 \rangle$ so the horizontal change is $\sqrt{1^2 + 1^2}$ which is $\sqrt{2}$. The vertical change is 1 times 1 plus 1 times 3 and umm is 4. So the slope or umm I mean the directional derivative is 4 over $\sqrt{2}$.

Student A1 seems to think of a directional derivative as a slope, as proposed in the GD. Like A1, 9 of the 11 students in the activity section related tangent plane to directional derivative (eight of them used slope). In the regular section, three of the 11 students constructed that relation; they all used a formula based on the gradient vector, like R1. This formula seems to have been used as an Action, a memorized procedure, since geometric understanding of the formula would require a Process of vertical change on a plane, which R1 did not give evidence of having constructed.

R1: It's the directional derivative of f at the point (1, 2) in the direction of vector. The magnitude of the vector is $\sqrt{1+1}$ which is $\sqrt{2}$. I know the directional derivative $Df_{\langle a,b \rangle}(x,y)$ where $\langle a,b \rangle$ is a unit vector, is equal to $a \cdot f_x(x,y) + b \cdot f_y(x,y)$, so the directional derivative is $Df_{\langle 1,1 \rangle}(1,2) = 1 \times \frac{1}{\sqrt{2}} + 3 \times \frac{1}{\sqrt{2}}$ which is $\frac{4}{\sqrt{2}}$.

DISCUSSION AND CONCLUSIONS

This study examines a second research cycle investigating students' understanding of the differential calculus of two-variable functions. The results of the first cycle suggested that the notions of tangent plane, total differential, and function remained isolated in the minds of most students. The results of the second cycle now show that it is possible to help students interrelate these notions based on the slope and local linearity approach. That is, by having students work collaboratively and discuss in class activities that use slope as a base to construct vertical change on a plane, and from there, exploring and interrelating the notions of tangent plane, total differential, function, and directional derivative in different representations. A contribution of this study is showing that students can succeed in this construction. The approach to the differential calculus of two-variable functions, based on the geometric understanding of slope and vertical change on a plane, is another contribution of this study.

The study's results suggest that students can obtain a deeper understanding of the differential calculus with this approach. In question 6a, we saw that slope can play a role in fomenting the interrelation of tangent plane, total differential, and function, notions which on the first research cycle seemed to remain isolated in the minds of most students. The results dealing with question 6b suggest that the construction of slope and vertical change on a plane, helps students relate tangent plane and total differential, thus showing an improvement on the results of the first research cycle (Trigueros et al., 2018), where no student showed to construct total differential as a Process. Directional derivative, as seen in question 6d, is another notion that students can relate to tangent plane, giving geometric meaning to the usual formula that students mostly tend to memorize, and thus understand as an Action conception. This is suggested by the study (Borji et al., 2023), which shows results about directional derivative that improved from the first research cycle, where only one of 26 students constructed a Process of directional derivative. All these differential calculus ideas are held together by the notions of slope and the derived vertical change on a plane, as suggested by the genetic decomposition and the results of the study.

The interview instrument in its entirety involves several components of the differential calculus of two-variable functions, including slope, function, vertical change on a plane, point-slopes equation of a plane, tangent plane, total differential, partial derivative, directional derivative, and gradient. Thus, the complexity of the corresponding Schema requires an investigation that takes advantage of tools like a GD stated in terms of the schema components and relations between components, the types of relations between Schema components, and the triad of stages of Schema development (Arnon et al., 2014). This is future work.

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